Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 58 (2008) 259–290
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Towards physically motivated proofs of the Poincaré and geometrization conjectures

Arkady L. Kholodenko*

375 H.L. Hunter Laboratories, Clemson University, Clemson, SC 29634-0973, USA

Received 11 January 2007; accepted 9 November 2007
Available online 17 November 2007

Abstract

Although the Poincaré and the geometrization conjectures were recently proved by Perelman, the proof relies heavily on properties of the Ricci flow previously investigated in great detail by Hamilton. Physical realization of such a flow can be found, for instance, in the work by Friedan [D. Friedan, Nonlinear models in $2+\varepsilon$ dimensions, Ann. Phys. 163 (1985) 318–419]. In his work the renormalization group flow for a nonlinear sigma model in $2+\varepsilon$ dimensions was obtained and studied. For $\varepsilon = 0$, by approximating the β-function for such a flow by the lowest order terms in the sigma model coupling constant, the equations for Ricci flow are obtained. In view of such an approximation, the existence of this type of flow in Nature is questionable. In this work, we find totally independent justification for the existence of Ricci flows in Nature. This is achieved by developing a new formalism extending the results of two-dimensional conformal field theories (CFT's) to three and higher dimensions. Equations describing critical dynamics of these CFT's are examples of the Yamabe and Ricci flows realizable in Nature. Although in the original works by Perelman some physically motivated arguments can be found, their role in his proof remain rather obscure. In this paper, steps are made toward making these arguments more explicit, thus creating an opportunity for developing alternative, more physically motivated, proofs of the Poincaré and geometrization conjectures.

MSC: primary 53B50; 53C21; 83C99; 83E30; 81T40; 82C27; secondary 82B27

Keywords: Ginzburg–Landau functional; Hilbert–Einstein action; Conformal field theories in two and three dimensions; Critical dynamics of phase transitions; Ricci and Yamabe flows; Dilaton gravity; Nash and Perelman's entropy

1. Introduction

The history of physics is full of examples of situations when experimental observations lead to very deep mathematical results. Beginning with Newton's dynamics equations, which gave birth to analysis, through the heat and wave equations, which gave birth to mathematical physics, through Maxwell's equations (experimentally discovered by Faraday), which ultimately led to special relativity on one hand and the theory of homology and cohomology on another, etc., thus stimulating the development of topology. Methods of celestial mechanics, especially those developed by Poincaré, led Bohr and his collaborators to quantum mechanics, on one hand, and to KAM and chaos

* Tel.: +1 864 656 5016; fax: +1 864 656 6613.
E-mail address: string@clemson.edu.

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.11.003

theory, on the other. A simple observation by Einstein that gravity can be locally eliminated in the freely falling elevator cabin led to general relativity, on one hand, and to great advancements in differential geometry, on the other. On the basis of these examples, it is only natural to expect that the recent complete proof of the Poincaré and geometrization conjectures of Perelman [1–3] should be connected with some processes taking place in the real world. The purpose of this work is to demonstrate that this is indeed the case. Since this paper is not a review, we made no efforts to list all the latest physics-related results inspired by Perelman's works. Instead, we shall only quote those works which are immediately relevant to the content of our paper.

We begin with the observation that in the work of Friedan [4] on the nonlinear sigma model in $2 + \varepsilon$ dimensions, published in 1985,[1] a reference was made to a talk by Bourguignon (delivered in 1979) "On Ricci curvature and the Einstein metric" in which for the first time the idea of usefulness of Ricci flows to differential geometry and related disciplines was introduced; e.g. see Ref. [10] in Friedan's paper. This idea was picked up by Hamilton in 1982 [5] who began from that time on a systematic study of Ricci flows. Although written with exceptional clarity, his results summarized in Ref. [6] apparently escaped physicists attention for many years. This is caused by several factors, which are worth mentioning. The Ricci-type flow equations appeared in the physics literature in the 80's in connection with the renormalization group analysis of nonlinear sigma models and also of strings propagating on curved backgrounds [4,7] without any reference to the parallel developments in mathematics. Because of this, the point of view was developed in the physics literature, especially that related to string theory [8], that all major results of general relativity along with quantum corrections can be deduced from the action functional whose minimization produces the Euler–Lagrange equations for the renormalization group (RG) flow for strings in curved backgrounds; e.g. see [8], ch. 3, page 180. In the simplest case such a functional (e.g. see Eq. (3.4.58) on page 180) is a modification of the Hilbert–Einstein functional describing gravity, leading to the action functional for the dilaton gravity. The only problem with such a functional lies in the fact that it is "living" in 26-dimensional space–time for a bosonic string and in 10-dimensional space–time for a fermionic (heterotic) string [9]. These facts apparently were known to Grisha Perelman who used them in his calculations, many of which (but not those involving such functionals) are specific just to three-dimensional Riemannian manifolds and, hence, are not readily extendable to higher dimensions and a pseudo-Riemannian type of metric.

At this point it is essential to notice that both in the work by Friedan [4] and in the string-theoretic literature just cited, the dilaton gravity functional emerges only as a result of the lowest order approximation in RG-type calculations. This fact can be seen very clearly on page 322 of Friedan's paper, where in his Eqs. (1.5) and (1.6) one has to put $\varepsilon = 0$ and retain only terms of order $O(T^0)$ in the sigma model coupling constant T^{-1}. Such a retention then straightforwardly leads to the equation for the Ricci flow.[2] When terms of order T and higher are included into these RG flow equations, such an RG flow is no longer of Ricci type. The procedure of an ad hoc truncation of an infinite asymptotic RG series expansion for the RG β-function (e.g. see Eq. (1.5) in Friedan's paper) in order to get the needed equation for the Ricci flow makes the physical existence of such a flow doubtful. Since both the Poincaré and the geometrization conjectures were proved with the help of the *unperturbed* (*not modified by terms other than those induced by diffeomorphisms*) Ricci flows [1–3], it is essential to find physical processes that employ such unperturbed Ricci-type flows. The present work provides examples of such physical processes. It enables us to reconnect recent significant mathematical advances with physical processes realizable in Nature in accord with the logic of development of mathematical physics described at the beginning of this section.

In view of space limitation, before discussing the content of this paper in some detail, we would like to mention the following. The existing proofs of the Poincaré and the geometrization conjectures are either too compressed [1] or too bulky [2,3,10]. In both cases, it may take a considerable amount of time for the nonexpert reader to learn and to understand these papers. Fortunately, there are succinct publications that present the key ideas of the proof in a language that is considerably more familiar to a much wider group of people with mathematical background. Among these reviews, we would especially recommend those written by Bessieres [11], on the proof of the Poincaré conjecture, and by Kapovich [12], on the proof of the geometrization conjecture. This paper is written under the assumption that our readers have previously read these reviews.

[1] It is based on his Ph.D. thesis completed in August of 1980.

[2] At the same time, obtaining the generating functional whose minimization produces the Euler–Lagrange equation for such a flow is nontrivial. In mathematics literature the credit for finding such a functional goes to Perelman. In physics literature this functional, Eq. (3.4.58) on page 180 of Ref. [8], was known much earlier. It describes the dilaton gravity (albeit in nonphysical space–time dimensions).

The organization of the rest of this paper is the following. In Section 2 the scaling analysis of the Ginzburg–Landau (G–L) functional, widely used in the theory of critical phenomena, is discussed using both arguments known in physics literature and those known in mathematics. In particular, we argue that in the physics literature there is a marked difference in treatments of two-dimensional and the higher dimensional models exhibiting critical behavior. While in two dimensions the full conformal invariance is used (at least at criticality), in dimensions higher than 2 only scale invariance is taken into account at criticality. Using scaling arguments from both physics and mathematics, we demonstrate how the existing scaling treatments in higher dimensions can be improved in order to take into account the effects of full conformal invariance. This improvement is brought to completion in Section 3 where we connect the G–L functional used in the physics literature with the Yamabe functional known in mathematics. We demonstrate that, if properly interpreted, both functionals upon minimization produce the same G–L-type equations as are used in literature on critical phenomena. The advantage of working with Yamabe functionals lies in our observations that (a) such a functional can be rewritten in terms of the Hilbert–Einstein action functional for Euclidean gravity and (b) such a functional is manifestly conformally invariant. Such a connection between the H–E action and its pseudo-Riemannian extension is needed in applications in relativity which are studied in Section 4. Connections between the G–L and Yamabe functionals exist only for dimensionality higher than 2. In two dimensions one has to find an analog of the G–L–Yamabe functional. This task is accomplished in Section 5 in which not only is such an analog found but also its connection with known results in string and CFT is obtained. Specifically, we were able to connect such an analog with the much earlier results obtained by Distler, Kawai and David for 2D quantum gravity in noncritical dimensions pedagogically summarized in the book by Hatfield [13]. Applications of their results to CFT are summarized in the lecture notes by Abdalla et al. [14] and, more recently, in the review by Nakayama [15]. Because of such connections, the extension of these two-dimensional CFT results to higher dimensions can be done quite systematically. This is discussed in Section 6. In Section 7 we develop an alternative method allowing us to reobtain results known already for two and higher dimensions. This formalism is based on the observation that results obtained in previous sections are just static solutions of more general equations describing critical dynamics for systems of the G–L type. By noticing that mathematically such dynamics coincides with that known for the Yamabe-type flows discovered by Hamilton [16], we follow the logic of his works in order to relate the Yamabe flow to the more general Ricci flow of which the Yamabe flow is a special case. To do so we use some results of Perelman [1], which he was using in his proofs of the Poincaré and the geometrization conjectures. Our use of Perelman's results is not mechanical however. By exposing physical arguments hidden in his work, we were able to reobtain many of his results much more simply as compared to treatments which can be found in current works by mathematicians [2,3,10] aimed at explaining and elaborating on many hidden details of Perelman's original proof. Finally, in Section 8 we discuss other possible physical systems whose dynamics is expected to be describable in terms of Ricci flows. We also discuss the conditions under which experiments on critical dynamics should be conducted in order to detect the effects of Ricci and Yamabe flows.

2. Scaling analysis of the G–L functional

Conventional scaling analysis of the G–L functional used routinely in physics literature can be found, for example, in Ref. [17]. This analysis differs somewhat from that for the ϕ^4 model as described in the monograph by Itzykson and Zuber [18]. For the sake of the discussion that will follow, we would like to provide a sketch of the arguments for both cases now.

We begin with the ϕ^4 model following Ref. [18]. Let $\mathcal{L}(x)$ be the Lagrangian of this scalar field model whose action functional in d dimensions $S[\phi]$ is given by $S[\phi] = \int \mathrm{d}^d x \mathcal{L}(x)$. Let furthermore λ be some nonnegative parameter. Then the requirement that $S[\phi]$ be independent of λ, i.e. $\int \mathrm{d}^d x \mathcal{L}(x) = \int \mathrm{d}^d x \lambda^d \mathcal{L}(\lambda x)$, leads to the constraint

$$\int \mathrm{d}^d x \left(x \cdot \frac{\partial}{\partial x} + d \right) \mathcal{L}(x) = 0 \tag{2.1}$$

obtained by differentiation of $S[\phi]$ with respect to λ with λ being set equal to 1 at the end of calculation. For $\mathcal{L}(x)$ given by

$$\mathcal{L}(x) = \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\hat{G}}{4!} \phi^4 \tag{2.2}$$

the change of $\mathcal{L}(x)$ under the infinitesimal scale transformation is given by[3]

$$\frac{\delta \mathcal{L}}{\delta \varepsilon} = \left(x \cdot \frac{\partial}{\partial x} + d \right) \mathcal{L}(x) + (d-4) \frac{\hat{G}}{4!} \phi^4 - m^2 \phi^2. \tag{2.3}$$

Comparison between Eqs. (2.1) and (2.3) implies that the action $S[\phi]$ is scale invariant if $d = 4$ and $m^2 = 0$. The result just obtained raises an immediate question: Given that the massless G–L action is scale invariant for $d = 4$, will it also be conformally invariant under the same conditions? We provide the answer to this question in several steps.

First, let M be some Riemannian manifold whose metric is g. Then, any metric $\tilde{g}$ conformal to g can be written as $\tilde{g} = \mathrm{e}^f g$ with f being a smooth real valued function on M [19]. Let Δ_g be the Laplacian associated with metric g[4] and, accordingly, let $\Delta_{\tilde{g}}$ be the Laplacian associated with metric $\tilde{g}$. Richardson [20] demonstrated that

$$\Delta_{\tilde{g}} = \mathrm{e}^{-f} \Delta_g - \frac{1}{2} \left(\frac{d}{2} - 1 \right) f \mathrm{e}^{-f} \Delta_g - \frac{1}{2} \left(\frac{d}{2} - 1 \right) \mathrm{e}^{-f} (\Delta_g f) + \frac{1}{2} \left(\frac{d}{2} - 1 \right) \mathrm{e}^{-f} \Delta_g \circ f \tag{2.4}$$

(here $(\Delta_g \circ f) \Psi$ should be understood as $\Delta_g (f(x) \Psi(x))$. Eqs. (2.3) and (2.4) imply that the conformally invariant (string-type) functional $S[\mathbf{X}] = \int_M \mathrm{d}^2 x \sqrt{g} (\nabla_g \mathbf{X}) \cdot (\nabla_g \mathbf{X})$ exists only for $d = 2$. For $d > 2$, in view of Eq. (2.4) the conformal invariance of the ϕ^4 model is destroyed. Thus, using conventional field-theoretic perturbational methods one encounters a problem with conformal invariance at the zeroth-order level (in the coupling constant $\hat{G}$), unless $d = 2$. This problem formally does not occur if one requires our physical model to be only scale invariant. This is undesirable however in view of the fact that in two dimensions an arbitrary conformal transformation of the metric tensor g of the underlying two-dimensional manifold M is permissible at criticality [21]. Abandoning the requirement of general conformal invariance in two dimensions in favor of scale invariance in higher dimensions would destroy all known string-theoretic methods of obtaining exact results in two dimensions. Our general understanding of critical phenomena depends crucially on our ability to solve two-dimensional models exactly. All critical properties for the same kinds of models in higher dimensions are expected to hold even in the absence of exact solvability. Fortunately, the situation can be considerably improved by reanalyzing and properly reinterpreting the scaling results for dimensions higher than 2.

This observation leads to the next step in our arguments. Using the book by Amit [17], we consider first the scaling of the noninteracting (free) G–L theory whose action functional is given by

$$S[\phi] = \int \mathrm{d}^d x \{ (\nabla \phi)^2 + m^2 \phi^2 \}. \tag{2.5}$$

Suppose now that, upon rescaling, the field ϕ transforms according to the rule[5]

$$\tilde{\phi}(Lx) = L^{\omega} \phi(x). \tag{2.6}$$

If we require

$$\int \mathrm{d}^d x \{ (\nabla \phi)^2 + m^2 \phi^2 \} = \int \mathrm{d}^d x L^d \{ (\tilde{\nabla} \tilde{\phi})^2 + \tilde{m}^2 \tilde{\phi}^2 \} \tag{2.7}$$

and use Eq. (2.6) we obtain

$$S[\phi] = \int \mathrm{d}^d x L^d \{ (\nabla \phi)^2 L^{2\omega - 2} + \tilde{m}^2 \phi^2 L^{2\omega} \}. \tag{2.8}$$

In order for the functional $S[\phi]$ to be scale invariant the mass m^2 should scale as follows: $\tilde{m}^2 = m^2 L^{-2}$. With this requirement the exponent ω is found to be $\omega = 1 - \frac{d}{2}$.

[3] In arriving at this result we took into account Eqs. (13)–(40) of Ref. [18] along with condition $D = \frac{d}{2} - 1$, with D defined in Eqs. (13)–(38) of the same reference. Also, we changed signs (as compared to the original source) in $\mathcal{L}(x)$ to be in accord with the accepted conventions for the G–L functional.

[4] That is $\Delta_g \Psi = -(\det g)^{-\frac{1}{2}} \partial_i (g^{ij} (\det g)^{\frac{1}{2}} \partial_j \Psi)$ for some scalar function $\Psi(x)$.

[5] Notice that this is already a special kind of conformal transformation.

The above scaling can be done a bit differently following the same Ref. [17]. For this purpose we notice that although the action $S[\phi]$ is scale invariant, there is some freedom of choice for the dimensionality of the field ϕ. For instance, instead of $S[\phi]$ we can consider

$$S[\phi] = \frac{1}{a^d} \int \mathrm{d}^d x \{(\nabla \phi)^2 + m^2 \phi^2\} \tag{2.9}$$

with a^d being some volume, say, $a^d = \int \mathrm{d}^d x$. Then, by repeating arguments related to Eq. (2.8), we obtain $\omega = 1$ (instead of $\omega = 1 - \frac{d}{2}$ previously obtained) in accord with Eq. (2-66) of the book by Amit [17]. Although, from the point of view of scaling analysis, the two results are actually equivalent, they become quite different if we want to extend such scaling analysis by considering general conformal transformations. Even though, in view of Eq. (2.4), such a task seems impossible to accomplish, fortunately, this is not true as we would like to demonstrate. This leads us to the next step.

Notice that the mass term scales as the scalar curvature R for some Riemannian manifold M, i.e. the scaling $\tilde{m}^2 = m^2 L^{-2}$ is *exactly the same* as the scaling of R given by

$$\tilde{R} = L^{-2} R. \tag{2.10}$$

This result can be found, for example, in the book by Wald, Ref. [22], Eq. (D.9), page 446. In general, the scalar curvature $R(g)$ changes under the conformal transformation $\hat{g} = \mathrm{e}^{2f} g$ according to the rule [19]

$$\hat{R}(\hat{g}) = \mathrm{e}^{-2f} \{R(g) - 2(d-1)\Delta_g f - (d-1)(d-2)|\nabla_g f|^2\} \tag{2.11}$$

where $\Delta_g f$ is the Laplacian of f and $\nabla_g f$ is the covariant derivative defined with respect to the metric g. From here we see that, indeed, for constant f's the scaling takes place in accord with Eq. (2.10). Now, however, we can do more. Following Lee and Parker [19], we simplify the above expression for R. For this purpose we introduce a substitution: $\mathrm{e}^{2f} = \varphi^{p-2}$, where $p = \frac{2d}{d-2}$, so that $\hat{g} = \varphi^{p-2} g$. With such a substitution, Eq. (2.11) acquires the following form:

$$\hat{R}(\hat{g}) = \varphi^{1-p}(\alpha \Delta_g \varphi + R(g)\varphi), \tag{2.12}$$

with $\alpha = 4\frac{d-1}{d-2}$. Clearly, such an expression makes sense only for $d \geq 3$ and breaks down for $d = 2$. But we know already the action $S[\mathbf{X}]$ which is both scale and conformally invariant for $d = 2$. It is given after Eq. (2.4). Fortunately, the results obtained in this section can be systematically extended in order to obtain actions which are both scale and conformally invariant in three dimensions and higher. This is described in the next section.

3. G–L functional and the Yamabe problem

We begin with the following observation. Let $\tilde{R}(\tilde{g})$ in Eq. (2.11) be some constant (that this is indeed the case we shall demonstrate shortly). Then Eq. (2.12) acquires the following form:

$$\alpha \Delta_g \varphi + R(g)\varphi = \hat{R}(\hat{g})\varphi^{p-1}. \tag{3.1}$$

By noticing that $p - 1 = \frac{d+2}{d-2}$ we obtain at once: $p - 1 = 3$ (for $d = 4$) and $p - 1 = 5$ (for $d = 3$). These are familiar Ginzburg–Landau values for the exponents of interaction terms for critical and tricritical G–L theories [23]. Once we recognize these facts, the action functional producing the G–L-type equation (3.1) can be readily constructed. For this purpose it is sufficient to rewrite Eq. (2.9) in a manifestly covariant form. We obtain

$$S[\varphi] = \frac{1}{\left(\int_M \mathrm{d}^d x \sqrt{g} \varphi^p\right)^{\frac{2}{p}}} \int_M \mathrm{d}^d x \sqrt{g} \{\alpha(\nabla_g \varphi)^2 + R(g)\varphi^2\} \equiv \frac{E[\varphi]}{\|\varphi\|_p^2}. \tag{3.2}$$

Minimization of this functional produces the following Euler–Lagrange equation:

$$\alpha \Delta_g \bar{\varphi} + R(g)\bar{\varphi} - \lambda \bar{\varphi}^{p-1} = 0 \tag{3.3}$$

with constant λ denoting the extremum value for the ratio:

$$\lambda = \frac{E[\bar{\varphi}]}{\|\bar{\varphi}\|_p^p} = \inf\{S[\varphi] : \hat{g} \text{ conformal to } g\}. \tag{3.4}$$

In accord with the Landau theory of phase transitions [24] it is expected that the conformal factor φ is a smooth nonnegative function on M achieving its extremum value $\bar{\varphi}$. Comparison between Eqs. (3.1) and (3.3) implies that $\lambda = \tilde{R}(\tilde{g})$ as required. These results belong to Yamabe, who obtained the Euler–Lagrange G–L-type equation (3.3) upon minimization of the functional $S[\varphi]$ without any knowledge of the Landau theory. The constant λ is known in the literature as the *Yamabe invariant* [19,25]. Its value is an invariant of the conformal class (M, g). Given these facts, we obtain the following:

Definition 3.1. The *Yamabe problem* lies in finding a compact Riemannian manifold (M, g) of dimension $n \geq 3$ whose metric is conformal to the metric $\hat{g}$ of a constant scalar curvature.

Subsequent developments, e.g. that given in Refs. [26,27], extended this problem to manifolds with boundaries and to noncompact manifolds. It is not too difficult to prove that the Yamabe–Ginzburg–Landau-like functional is manifestly conformally invariant. For this purpose, we need to rewrite Eq. (2.12) in the following equivalent form:

$$\varphi^p \hat{R}(\hat{g}) = (\alpha\varphi\Delta_g\varphi + R(g)\varphi^2). \tag{3.5}$$

This can be used in order to rewrite $E[\varphi]$ as follows: $E[\varphi] = \int \mathrm{d}^d x\sqrt{g}\varphi^p \hat{R}(\hat{g})$. Next, by noting that $\int \mathrm{d}^d x\sqrt{\hat{g}} = \int \mathrm{d}^d x\sqrt{g}\varphi^p$, we can rewrite the Yamabe functional in the Hilbert–Einstein form

$$S[\varphi] = \frac{\int \mathrm{d}^d x\sqrt{\hat{g}}\hat{R}(\hat{g})}{\left(\int \mathrm{d}^d x\sqrt{\hat{g}}\right)^{\frac{2}{p}}}, \tag{3.6}$$

where both the numerator and the denominator are invariant with respect to conformal changes in the metric, e.g. scale change.

Remark 3.2. In order to use these results in statistical mechanics, we have to demonstrate that the extremum of the Yamabe functional $S[\varphi]$ is realized for manifolds M whose scalar curvature $R(g)$ in Eq. (3.3) is also constant. This is the essence of the Yamabe problem (as defined above) for physical applications. Fortunately, the Yamabe problem has been solved positively by several authors starting with Yamabe himself (whose proof contained some mistakes however) and culminated in the work by Schoen [28]. Details can be found in the review paper by Lee and Parker [19] and in the monograph by Aubin [29].

Remark 3.3. In view of the relation $\int \mathrm{d}^d x\sqrt{\hat{g}} = \int \mathrm{d}^d x\sqrt{g}\varphi^p$, it is clear that for the fixed background metric g, Eq. (3.3) can be obtained alternatively using the following variational functional:

$$\tilde{S}[\varphi] = \int \mathrm{d}^d x\sqrt{g}\{\alpha(\nabla_g\varphi)^2 + R(g)\varphi^2\} - \tilde{\lambda}\int \mathrm{d}^d x\sqrt{g}\varphi^p \tag{3.7}$$

where the Lagrange multiplier $\tilde{\lambda}$[6] is responsible for the volume constraint. Such a form of the functional $\tilde{S}[\varphi]$ brings this higher dimensional result into accord with that to be developed below for two dimensions (e.g. see Section 5, Eq. (5.24), and Section 7).

Apart from the normalizing denominator, Eq. (3.6) represents the Hilbert–Einstein action for pure gravity defined for Riemannian d-dimensional space. The denominator, the volume V taken to power $\frac{2}{p}$, serves the purpose of making $S[\varphi]$ manifestly conformally invariant; see Ref. [28], page 150.

[6] $\tilde{\lambda} = \lambda\frac{2}{p} \equiv \lambda\frac{d-2}{d}$.

4. From the G–L to the Hilbert–Einstein functional

In this section we would like to analyze the significance of the cosmological constant term in the Hilbert–Einstein action for gravity from the point of view of the G–L model. For this purpose, following Dirac [30] let us consider the extended Hilbert–Einstein (H–E) action functional $S^c(g)$ for pure gravity with an extra (cosmological constant) term defined for some (pseudo-)Riemannian manifold M (of total space–time dimension d) without a boundary:

$$S^c(g) = \int_M R\sqrt{g}\mathrm{d}^d x + C\int_M \mathrm{d}^d x\sqrt{g}. \tag{4.1}$$

The (cosmological) constant C is determined on the basis of the following chain of arguments. First, we introduce the following.

Definition 4.1. Let R_{ij} be the Ricci curvature tensor. Then the *Einstein space* is defined as a solution of the following vacuum Einstein equation:

$$R_{ij} = \lambda g_{ij} \tag{4.2}$$

with λ being constant.

From this definition it follows that in both the Riemannian and pseudo-Riemannian cases

$$R = d\lambda. \tag{4.3}$$

Following Dirac [30], variation of the action $S^c(g)$ produces

$$R_{ij} - \frac{1}{2}g_{ij}R + \frac{1}{2}Cg_{ij} = 0. \tag{4.4}$$

The combined use of Eqs. (4.3) and (4.4) produces

$$C = \lambda\left(\frac{d-2}{2}\right). \tag{4.5}$$

With these results, using Eqs. (4.3) and (4.5) we can rewrite Eq. (4.4) as follows:

$$R_{ij} - \frac{1}{2}g_{ij}R + \frac{1}{2d}(d-2)Rg_{ij} = 0. \tag{4.6}$$

Remark 4.2. These observations allow us to look at possible pseudo-Riemannian extension of the results originally obtained by Yamabe, Refs. [19,29,31], for Riemannian manifolds. Although at first sight this might seem of significant importance for high energy physics applications, in a companion publication [32] we shall argue that dealing with Riemannian manifolds is already quite sufficient so that the Yamabe results can be used in high energy physics without changes. It should be clear as well that the earlier introduced notion of the Einstein space is applicable to both Riemannian and pseudo-Riemannian manifolds.

In view of this remark, we would like to argue that Eq. (4.6) can also be obtained by varying the Yamabe functional, Eq. (3.6). Indeed, following Aubin [29] and Schoen [28], let t be some small parameter labeling the family of metrics: $g_{ij}(t) = g_{ij} + th_{ij}$. Then, these authors demonstrate that

$$\left(\frac{\mathrm{d}R_t}{\mathrm{d}t}\right)_{t=0} = \nabla^i\nabla^j h_{ij} - \nabla^j\nabla_j h^i_i - R^{ij}h_{ij} \tag{4.7}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{|g_t|}\right)_{t=0} = \frac{1}{2}\sqrt{|g|}g^{ij}h_{ij}, \tag{4.8}$$

where, as usual, $|g| = \left|\det g_{ij}\right|$. Consider now the Yamabe functional, Eq. (3.6), but, this time, written for the family of metrics which belong to the same conformal class. We have

$$\mathcal{R}(g(t)) = (V(t))^{\frac{-2}{p}} \int_M R(g(t)) DV(t), \tag{4.9}$$

where the volume is given by $V(t) = \int_M \mathrm{d}^d x \sqrt{g(t)}$ and, accordingly, $DV(t) = \mathrm{d}^d x \sqrt{g(t)}$. Using Eqs. (4.7) and (4.8) in Eq. (4.9) and taking into account that the combination $\nabla^i \nabla^j h_{ij} - \nabla^j \nabla_j h_i^i$ is the total divergence produces the following result:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}(g(t))\right)_{t=0} = V(0)^{-\frac{2-d}{d}} \left[\int_M (Rg^{ij}/2 - R^{ij}) h_{ij} DV(0) \int_M DV(0) - \left(\frac{1}{2} - \frac{1}{d}\right) \int_M DV(0) \int_M h_{ij} g^{ij} DV(0)\right]. \tag{4.10}$$

If the metric g is the critical point of $\mathcal{R}(g(t))$, then

$$\left[R_{ij} - \frac{R}{2} g_{ij}\right] \int_M DV(0) + \left(\frac{1}{2} - \frac{1}{d}\right) \left(\int_M R DV(0)\right) g_{ij} = 0. \tag{4.11}$$

From here, multiplication of both sides by g^{ij} and subsequent summation produces at once

$$R - \frac{R}{2} d + \left(\frac{1}{2} - \frac{1}{d}\right) \langle R \rangle d = 0, \tag{4.12}$$

where $\langle R \rangle = \frac{1}{V(0)} \int R DV(0)$ is the average scalar curvature. Eq. (4.12) can be rewritten as $R = \langle R \rangle$. But this condition is *formally* equivalent to the Einstein condition, Eq. (4.2), in view of Eqs. (4.3), (4.6)! Hence, *under such circumstances,* Eqs. (4.11) and (4.12) are equivalent.

Remark 4.3. It is important to mention that in general the condition $R = \langle R \rangle$ is more restrictive than the condition, Eq. (4.2), defining Einstein spaces. That is to say, the constant Ricci tensor R_{ij} always leads to the constant scalar curvature R while the Riemannian spaces of constant scalar curvature R are not necessarily spaces for which the Einstein condition, Eq. (4.2), holds. Hence, they may or may not be of Einstein type. This was demonstrated by Herglotz in 1916 (immediately after Einstein's general relativity was completed) [33, page 148]. His results were subsequently generalized in Refs. [34,35]. We shall return to this topic in Section 7 where we discuss this problem dynamically following the works by Hamilton and Perelman. Using dynamical arguments, we demonstrate that the equation $R = \langle R \rangle$ is compatible with the Einstein condition and, hence, describes the Einstein spaces.

In view of this remark, in the rest of this work we shall develop our formalism under the assumption that for the set of metrics of fixed conformal class, the variational problem for the G–L functional, Eq. (3.7), is equivalent to the variational problem for the H–E functional, Eq. (4.1), for pure gravity in the presence of the cosmological constant. Because of this noted formal equivalence, the presence of the cosmological term in the H–E action amounts to volume conservation for the G–L variational problem. From here, it should be clear that Eq. (2.12) is equivalent to the condition $R = \langle R \rangle$ in view of Eq. (3.4).

The result shown in Eq. (4.12) becomes trivial for $d = 2$. Physically, however, the case $d = 2$ is important since it is relevant to all known exactly solvable models of statistical mechanics treatable by methods of conformal field theories. Hence, now we would like to discuss needed modifications of the results obtained in order to obtain a two-dimensional analog of the G–L theory.

5. Ginzburg–Landau-like theory in two dimensions

5.1. Designing the two-dimensional G–L–Yamabe functional

From the field-theoretic treatments of the G–L model [36] we know that a straightforward analysis based on asymptotic ε-expansions from the critical dimension (4) to the target dimension (2) is impractical. At the same time, the results of CFT and exactly solvable models make sense thus far only for $d = 2$. The question arises: Is there an analog of G–L equation (3.1) in two dimensions? And, if such an analog does exist, what use can be made of such a result? In this section we provide affirmative answers to these questions. We demonstrate that: (a) indeed, such a two-dimensional analog of the G–L equation does exist and is given by the Liouville equation (5.19), (b) the functional, Eq. (5.14), whose minimization produces such an equation is the exact two-dimensional analog of the G–L–Yamabe functional, Eq. (3.2), (c) these results can be (re)obtained from the existing string-theoretic formulations of the CFT which were developed entirely independently.

To discuss topics related to items (a) and (b) just mentioned, we begin with the observation that in two dimensions, Eq. (2.4) acquires a very simple form:

$$\Delta_{\hat{g}} = \mathrm{e}^{-2f} \Delta_g, \tag{5.1}$$

where we use a factor of 2 to be in accord with Eq. (2.11) for the scalar curvature. According to Eq. (2.11), the scalar curvature in two dimensions transforms like

$$\hat{R}(\hat{g}) = \mathrm{e}^{-2f}\{R(g) - \Delta_g 2f\} \tag{5.2}$$

while the area $\mathrm{d}A = \mathrm{d}^2x\sqrt{g}$ transforms like

$$\mathrm{d}\hat{A} = \mathrm{e}^{-2f}\mathrm{d}A. \tag{5.3}$$

These facts immediately suggest that the previously introduced action functional

$$S[\mathbf{X}] = \int_M \mathrm{d}^2x\sqrt{g}(\nabla_g \mathbf{X}) \cdot (\nabla_g \mathbf{X}) \tag{5.4}$$

is conformally invariant. Using results of Polyakov [37] and noting that $(\nabla_g \mathbf{X}) \cdot (\nabla_g \mathbf{X}) = g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu$, we need to consider the following path integral[7]:

$$\exp(-\mathcal{F}(g)) \equiv \int D[\phi] \exp\left(-\frac{1}{2}\int_M \mathrm{d}^2x\sqrt{g}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right). \tag{5.5}$$

Here the symbol $\mathcal{F}(g)$ stands for the "free energy" usually defined this way in statistical mechanics.[8] Fortunately, this integral was calculated by Polyakov [37] for two-dimensional manifolds M *without* boundaries and by Alvarez [38] for manifolds *with* boundaries. In this work, to avoid unnecessary complications, we shall be concerned with manifolds *without* boundaries. Although in the original work by Polyakov, one can find the final result of the calculation of the above path integral, the details of this calculation can only be found elsewhere. In particular, we shall follow pedagogically written papers by Weisberger [39,40] and Osgood, Phillips and Sarnak [41–43] (OPS).

To begin, let $\hat{g}_{\alpha\beta}$ be some reference metric, while $g_{\alpha\beta}$ is a metric conformally related to it, i.e. $g_{\alpha\beta} = \exp(-2\varphi)\hat{g}_{\alpha\beta}$. Should the above path integral be for the flat (i.e. $g_{\alpha\beta} = \delta_{\alpha\beta}$) two-dimensional manifold, one would have at once the result: $\mathcal{F} = \frac{1}{2}\ln\det'\Delta_0$, where the prime indicates that the zero mode is omitted. Because the flatness assumption in general is physically incorrect, it is appropriate to pose a problem: How is the path integral for the metric g related to the path integral for the metric $\hat{g}$? The paper [41] by OPS provides an answer; e.g. see Eq. (1.13) of this reference. To connect this equation with that known in the physics literature we replace it by the equivalent expression

$$\ln\left(\frac{\det'\Delta_{\hat{g}}}{A_{\hat{g}}}\right) - \ln\left(\frac{\det'\Delta_g}{A_g}\right) = -\frac{1}{6\pi}\left[\frac{1}{2}\int_M \mathrm{d}A_g\{|\nabla_g\varphi|^2 + R(g)\varphi\}\right] \tag{5.6}$$

useful in applications to strings and CFT; e.g. see Ref. [13], page 637.[9]

[7] Without loss of generality, we would like to consider the case of one component field ϕ only.

[8] Usually, instead of $\mathcal{F}(g)$ one writes $\mathcal{F}(g)/k_BT$, where T is the temperature and k_B is Boltzmann's constant. In the present case the problems that we are studying do not require specific values for these constants. For this reason they will be suppressed.

[9] In arriving at this result we took into account that the Gaussian curvature K in Ref. [41] is related to Riemannian curvature R as $R = 2K$.

It is worthwhile to provide a few computational details leading to Eq. (5.6). Let, therefore, $\hat{\psi}_i$ be eigenfunctions of the Laplacian Δ_g with eigenvalues λ_i arranged in such a way that $0 = \hat{\lambda}_0 < \hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \cdots$, i.e.

$$-\Delta_g \hat{\psi}_i + \hat{\lambda}\hat{\psi}_i = 0. \tag{5.7}$$

Then, we construct the zeta function

$$\zeta_g(s) = \sum_{i=1}^{\infty} \hat{\lambda}_i^{-s} \tag{5.8}$$

in such a way that

$$\det' \Delta_g = \exp(-\zeta_g'(0)) \tag{5.9}$$

with $\zeta_g'(0) = \left(\frac{\mathrm{d}}{\mathrm{d}s}\zeta_g(s)\right)_{s=0}$. Using Eq. (5.1), we obtain as well

$$-\mathrm{e}^{-2\varphi}\Delta_g \psi_i + \lambda\psi_i = 0. \tag{5.10}$$

In particular, for a constant $\varphi = \bar{\varphi}$ we obtain

$$\zeta_{\hat{g}}(s) = \sum_{i=1}^{\infty} \left(\mathrm{e}^{-2\bar{\varphi}}\hat{\lambda}_i\right)^{-s} = \mathrm{e}^{2s\bar{\varphi}}\zeta_g(s). \tag{5.11}$$

Use of Eq. (5.9) in Eq. (5.11) produces

$$\zeta_{\hat{g}}'(0) = \zeta_g'(0) + 2\bar{\varphi}\left(\frac{\chi(M)}{6} - 1\right). \tag{5.12}$$

This result was obtained with help of the known fact, Ref. [41], Eq. (1.9), that

$$\zeta_g(0) = \frac{\chi(M)}{6} - 1 \tag{5.13}$$

with $\chi(M)$ being Euler's characteristic of a two-dimensional manifold M without boundaries. In view of the definition, Eq. (5.9), and the Gauss–Bonnet theorem, we observe that Eq. (5.6) is reduced to Eq. (5.12) for the case of constant conformal factor $\varphi = \bar{\varphi}$, as required.

Using Eq. (5.6) and, following OPS [41], we would like to consider the related functional $\mathcal{F}(\varphi)$ defined by

$$\mathcal{F}(\varphi) = \frac{1}{2}\int_M \mathrm{d}A_g\{|\nabla_g \varphi|^2 + R(g)\varphi\} - \pi\chi(M)\ln\int_M \mathrm{d}A_g \mathrm{e}^{2\varphi}. \tag{5.14}$$

This functional is the exact analog of the Yamabe functional, Eq. (3.2), in dimensions 3 and higher as we shall demonstrate below and in Section 7. In the meantime, in view of Eq. (5.6), it can be rewritten in the following equivalent form[10]:

$$\mathcal{F}(\varphi) = -6\pi \ln\det' \Delta_g + \pi(6 - \chi(M))\ln A. \tag{5.15}$$

Let a be some constant, then

$$\mathcal{F}(\varphi + a) = \mathcal{F}(\varphi). \tag{5.16}$$

Such invariance signifies the fact that the above action is scale invariant. This property is in complete accord with Eqs. (2.7) and (2.8). If we impose a constraint (fix the gauge): $A = 1$, then we end up with the Liouville-like action used in CFT. We would like to explain all of this in some detail now.

[10] Here one should understand the word "equivalence" in the sense that the two functionals produce the same critical metrics upon minimization.

Following OPS, it is convenient to replace the constraint $A = 1$ by the alternative constraint on the field φ:

$$\int_M \varphi \mathrm{d}A_g = 0. \tag{5.17}$$

To demonstrate that such a constraint is equivalent to the requirement $A = 1$, we note that, provided that the field ψ minimizes $\mathcal{F}(\psi)$ subject to the constraint Eq. (5.17), the field

$$\varphi = \psi - \frac{1}{2} \ln \int_M \exp(2\psi) \mathrm{d}A_g \tag{5.18}$$

minimizes $\mathcal{F}(\varphi)$ subject to the constraint $A = 1$. Using these facts, minimization of $\mathcal{F}(\psi)$ produces the following Liouville equation:

$$-\Delta_g \psi + \frac{1}{2} R(g) - \frac{2\pi \chi(M) \exp(2\psi)}{\int_M \exp(2\psi) \mathrm{d}A_g} = 0. \tag{5.19}$$

Comparing this result with Eq. (5.2) and taking into account that

$$\frac{2\pi \chi(M)}{\int_M \exp(2\psi) \mathrm{d}A_g} = \hat{R}(\hat{g}) = \mathrm{const} \tag{5.20}$$

we conclude that, provided that the background metric g is given so that the scalar curvature $R(g)$ (not necessarily constant) can be calculated, the Liouville equation (5.19) is exactly analogous to that previously obtained in Eq. (3.3) (which is the same as Eq. (2.12)). In view of this analogy, use of Eqs. (3.4) and (3.6) as well as Eqs. (5.14) and (5.15) causes the functional $\mathcal{F}(\varphi)$ to attain its extremum for metric $\hat{g}$ of constant scalar curvature $\hat{R}(\hat{g})$. To decide whether the extremum is minimum or maximum we have to consider separately the cases $\chi(M) > 0$ and $\chi(M) \leq 0$. In the case of $\chi(M) > 0$ we have only to consider manifolds homeomorphic to S^2 so that $\chi(M) = 2$. Fortunately, this case was considered in detail by Onofri [44].[11] Using his work, the following inequality:

$$\ln \int_{S^2} \mathrm{d}A_{\hat{g}} \exp(\psi) \leq \int_{S^2} \mathrm{d}A_{\hat{g}} \psi + \frac{1}{4} \int_{S^2} \mathrm{d}A_{\hat{g}} |\nabla \psi|^2 \tag{5.21}$$

attributed to Aubin [29] and inspired by earlier results by Trudinger and Moser [45] is helpful for deciding whether the extremum obtained is a minimum or maximum. Here $\hat{g}$ is the metric of the unit sphere S^2 with a constant Gaussian curvature of 1. The metric g conformal to $\hat{g}$ is given by $g = \exp(2\psi)\hat{g}$, with ψ obeying the Liouville equation (just like Eq. (5.19)), where both $R(g)$ and $R(\hat{g})$ are constant by virtue of the initial choice of $\hat{g}$. By combining Eqs. (5.6), (5.14) and (5.15) with the inequality (5.21) and taking into account that by design $\ln A_{\hat{g}} = 0$, we obtain

$$-3\pi \ln \frac{\det' \Delta_g}{\det' \Delta_{\hat{g}}} = \frac{1}{4} \int_{S^2} \mathrm{d}A_{\hat{g}} \{|\nabla_{\hat{g}} \psi|^2 + 2\psi\} - \ln \int_{S^2} \mathrm{d}A_{\hat{g}} \exp(2\psi) \geq 0, \tag{5.22}$$

with equality occurring only at the extremum $\psi = \psi^*$, with function ψ^* being a solution of the Liouville equation (5.19). It can be shown [44] that: (a) such a solution involves only Möbius transformations of the sphere S^2 and (b) the functional $\mathcal{F}(\varphi)$ is invariant with respect to such transformations. The case of $\chi(M) \leq 0$ is treated in Section 2.2 of the OPS paper, Ref. [41], and leads to the same conclusions about the extremality of the functional $\mathcal{F}(\varphi)$.

Corollary 5.1. *The two-dimensional results just obtained* by design *are in accord with results obtained in higher dimensions (discussed in Sections* 3 *and* 6 *and to be discussed in Section* 7*). In particular,* the functional $\mathcal{F}(\varphi)$ is the exact analog of the Yamabe functional $S[\varphi]$, *Eq.* (3.2)*. Since in both cases the functionals are "translationally" (actually, "scale") invariant, e.g. compare Eq.* (3.6) *with Eq.* (5.16)*, in both cases the extremum is realized for the metric conformal to the metric of constant scalar curvature; e.g. compare Eq.* (3.3) *with the Liouville equation* (5.19)*.*

[11] Generalization of these two-dimensional results to higher dimensional manifolds of even dimensions can be found in the paper by Beckner [45].

5.2. Connections with string and CFT

The results obtained allow us now to discuss topic (c) listed at the beginning of this section. Using known results for string and conformal field theories, the results just obtained can be given a statistical mechanical interpretation. Following [14,15,46], it is of interest to consider averages of the vertex operators. They are defined by

$$\left\langle \prod_{i=1}^{n} \exp(\beta_i \phi(z_i)) \right\rangle \equiv \int D[\phi] \exp\{-S_L(\phi)\} \prod_{i=1}^{n} \exp(\beta_i \phi(z_i)), \tag{5.23}$$

where the Liouville action $S_L(\phi)$ is given (in notation adopted from these references) by

$$S_L(\phi) = \frac{1}{8\pi} \int_M \mathrm{d}A_{\hat{g}} [|\nabla\phi|^2 - QR(\hat{g})\phi + 8\pi\bar{\mu}\exp(\alpha_+\phi)]. \tag{5.24}$$

The actual values and the meaning of the constants Q, $\bar{\mu}$ and α_+ are explained in these references and are of no immediate use for us. Clearly, upon proper rescaling, we can bring $S_L(\phi)$ to the form which agrees with $\mathcal{F}(\varphi)$, defined by Eq. (5.14), especially in the trivial case when both $\chi(M)$ and $\bar{\mu}$ are zero. When they are not zero, the situation in the present case becomes totally analogous to that discussed earlier for the Yamabe functional. In particular, in Section 3 we noted that the G–L Euler–Lagrange equation (3.3) can be obtained either by minimization of the Yamabe functional, Eq. (3.2) (or (4.9)), or by minimization of the G–L functional, Eq. (3.7), where the coupling constant λ plays the role of the Lagrange multiplier keeping track of the volume constraint. In the present case, variation of the Liouville action $S_L(\phi)$ will produce the Liouville equation, e.g. see Eqs. (5.19) and (5.20), which is the two-dimensional analog of the G–L equation (3.3). Such variation is premature, however, since we can reobtain $\mathcal{F}(\varphi)$ exactly using the path integral, Eq. (5.23). This procedure then will lead us directly to the Liouville equation (5.19).

For this purpose, we need to consider the path integral, Eq. (5.23), in the absence of sources, i.e. when all $\beta_i = 0$. Following ideas of Refs. [14,15,46], we take into account that: (a)

$$\frac{1}{4\pi} \int_M \mathrm{d}A_{\hat{g}} R(\hat{g}) = \chi(M) = 2 - 2g \tag{5.25}$$

with g being genus of M and (b) the field ϕ can be decomposed into $\phi = \phi_0 + \varphi$ in such a way that ϕ_0 is coordinate-independent and φ is subject to the constraint given by Eq. (5.17). Then, use of the identity

$$\int_{-\infty}^{\infty} \mathrm{d}x \exp(ax) \exp(-b\exp(\gamma x)) = \frac{1}{\gamma} b^{-\frac{a}{\gamma}} \Gamma\left(\frac{a}{\gamma}\right) \tag{5.26}$$

(with $\Gamma(x)$ being Euler's gamma function) in the path integral, Eq. (5.23), requires us to evaluate the following integral:

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \mathrm{d}\phi_0 \exp\left(\phi_0 \frac{Q}{2} \chi(M)\right) \exp\left(\bar{\mu} \int_M \mathrm{d}A_{\hat{g}} \exp(\alpha_+\varphi)\right) \exp(\alpha_+\phi_0) \\ &= \frac{\Gamma(-s)}{\alpha_+} \left(\bar{\mu} \int_M \mathrm{d}A_{\hat{g}} \exp(\alpha_+\varphi)\right)^s \end{aligned} \tag{5.27}$$

with s being given by

$$s = -\frac{Q}{2\alpha_+} \chi(M). \tag{5.28}$$

Using this result in Eq. (5.23), we obtain the following path integral (up to a constant):

$$Z[\varphi] = \int D[\varphi] \exp(-\hat{\mathcal{F}}(\varphi)) \tag{5.29}$$

with functional $\hat{\mathcal{F}}(\varphi)$ given by

$$\hat{\mathcal{F}}(\varphi) = S_L(\varphi; \bar{\mu} = 0) - \frac{Q}{2\alpha_+} \chi(M) \ln\left[\bar{\mu} \int_M \mathrm{d}A_{\hat{g}} \exp(\alpha_+\varphi)\right]. \tag{5.30}$$

This functional (up to rescaling of the field φ) is just the same as the Yamabe-like functional $\mathcal{F}(\varphi)$ given by Eq. (5.14).

Define now the free energy $\mathcal{F}$ in the usual way via $\mathcal{F} = -\ln Z[\varphi]$ (as was done after Eq. (5.5)) and consider the saddle point approximation to the functional integral $Z[\varphi]$. Then, for spherical topology in view of Eq. (5.22), we (re)obtain $\mathcal{F} \geq 0$ with equality obtained exactly when $\varphi = \psi^*$. Inclusion of the sources (or the vertex operators) can be taken into account also, especially in view of the results of [43].

To understand better the physical significance of the results obtained, it is useful to reobtain them using a somewhat different method. The results obtained with this alternative method are also helpful when we discuss their higher dimensional analogs in the next section. For this purpose we write (up to a normalization constant)

$$\int D\left[\phi\right] \exp\{-S_L(\phi)\} = \int_0^\infty \mathrm{d}A \mathrm{e}^{-\bar{\mu}A} Z_L(A), \tag{5.31}$$

where

$$Z_L(A) = \int D[\phi]\delta\left(\int_M \mathrm{d}A_{\hat{g}} \exp(\alpha_+\phi) - A\right) \exp(S_L(\phi; \bar{\mu} = 0)). \tag{5.32}$$

If, as before, we assume that $\phi = \phi_0 + \varphi$, then an elementary integration over ϕ_0 produces the following explicit result for $Z_L(A)$:

$$Z_L(A) = \frac{-1}{\alpha_+} A^\omega \int D[\varphi] \left[\int_M \mathrm{d}A_{\hat{g}} \exp(\alpha_+\varphi)\right]^{-(\omega+1)} \exp(-S_L(\varphi; \bar{\mu} = 0)), \tag{5.33}$$

where the exponent ω is given by

$$\omega = \frac{\chi(M)Q}{2\alpha_+} - 1. \tag{5.34}$$

Finally, using Eq. (5.33) in Eq. (5.31) produces Eqs. (5.29) and (5.30) (again, up to a constant factor). An overall "−" sign can be removed by proper normalization of the path integral. These results can be used for computation of the correlation functions of conformal field theories (CFT). Details can be found in [14,15,46].

6. Designing higher dimensional CFT(s)

6.1. General remarks

In the previous section we explained a delicate interrelationship between the path integrals Eqs. (5.5) and (5.23). From the literature on CFT cited earlier it is known that, actually, *both* are being used for the design of different CFT models in two dimensions. For instance, if one entirely ignores the effects of curvature in Eq. (5.5), then one ends up with the Gaussian-type path integral whose calculation for a flat torus is discussed in detail in Ref. [21], pages 340–343, and, by different methods, in our Ref. [47]. By making appropriate changes of the boundary conditions in such a path integral (or, equivalently, by considering appropriately chosen linear combinations of modular invariants) it is possible to build partition functions for all existing CFT models. For the same purpose one can use the path integral given by Eq. (5.23) but the calculation proceeds differently as explained in Section 5. Since in two dimensions the conformal invariance is crucial in obtaining exact results, use of the Gaussian-type path integrals is, strictly speaking, not permissible. Fortunately, the saddle point-type calculations made for the path integral, Eq. (5.23), produce the same results since the extremal metrics happen to be flat [47]. If one does not neglect curvature effects in Eq. (5.5), one eventually ends up with the integrand of the path integral, Eq. (5.23). This result is a consequence of Eq. (5.6) known as the conformal anomaly. If one would like to proceed *in an analogous fashion* in dimensions higher than 2 one should be aware that there is a profound difference between calculations done in odd and even dimensions. We would like to explain this circumstance in some detail now.

In two dimensions the conformal invariance of the action, Eq. (5.4), has been assured by the transformational properties of the two-dimensional Laplacian given by Eq. (5.1). In higher dimensions, the Laplacian is transformed according to Eq. (2.4), so that even the simplest Gaussian model is not conformally invariant! This observation makes use of traditional string-theoretic methods in higher dimensions problematic. In two dimensions these are based on a two-stage process: first one calculates the path integral, Eq. (5.5), exactly and, second, one uses the result of such a calculation (the conformal anomaly) as an input in another path integral, e.g. Eq. (5.23), which is obtained by integrating this input over all members of the conformal class. Since in odd dimensions there is no conformal anomaly as we shall demonstrate shortly, such a two-stage process cannot be used. The situation can be improved considerably if we *do not* rely on the use of the two-stage process just described. We would like to explain this fact in some detail now.

Even though the transformational properties (with respect to conformal transformations) of the Laplacian, Eq. (2.4), in dimensions higher than 2 are rather unpleasant, fortunately, they can be considerably improved if, instead of the usual Laplacian, one uses the conformal (Yamabe) Laplacian $\Box_g$ defined by

$$\Box_g = \Delta_g + \hat{\alpha} R(g), \tag{6.1}$$

where $\hat{\alpha} = \alpha^{-1} = \frac{1}{4}\frac{d-2}{d-1}$. By construction, in two dimensions it becomes the usual Laplacian. In higher dimensions its transformational properties are much simpler than those for the usual Laplacian (e.g. see Eq. (2.4)). Indeed, it can be shown [48] that

$$\Box_{e^{2f}g} = e^{-(\frac{d}{2}+1)f}\Box_g e^{(\frac{d}{2}-1)f}. \tag{6.2a}$$

This result can be easily understood if we use the results of Section 2. Indeed, since $e^{2f} = \varphi^{p-2}$ and since $p = \frac{2d}{d-2}$, we obtain at once $e^{(\frac{d}{2}-1)f} = \varphi$, while the factor $e^{-(\frac{d}{2}+1)f}$ is transformed into φ^{1-p}. From here, it is clear that Eq. (2.12) for scalar curvature can be equivalently rewritten as

$$\hat{R}(\hat{g}) = \alpha\varphi^{1-p}(\Delta_g\varphi + \alpha^{-1}R(g)\varphi), \tag{6.3}$$

where $\hat{g} = e^{2f}g$, so that

$$\hat{R}(\hat{g}) = \alpha\Box_{e^{2f}g}. \tag{6.4}$$

In practical applications it could be more useful to consider two successive conformal transformations made with the conformal factors e^{2f} and e^{2h}. If $e^{2f} = \varphi^{p-2}$ and $e^{2h} = \psi^{p-2}$ then we obtain

$$\Delta_{\hat{g}}\psi + \alpha^{-1}R(\hat{g})\psi = \varphi^{1-p}\left[\Delta_g(\varphi\psi) + \alpha^{-1}R(g)(\varphi\psi)\right]. \tag{6.2b}$$

This result is, of course, equivalent to Eq. (6.2a).

Consider now the following path integral:

$$\exp(-\mathcal{F}(g)) = \int D[\varphi]\exp\{-S_{\Box_g}(\varphi)\}, \tag{6.5}$$

where

$$S_{\Box_g}(\varphi) = \alpha\int_M d^dx\sqrt{g}\{(\nabla_g\varphi)^2 + \alpha^{-1}R(g)\varphi^2\}$$
$$= \alpha\int_M d^dx\sqrt{g}[\varphi\Box_g\varphi] = \int_M d^dx\sqrt{\tilde{g}}\hat{R}(\hat{g}). \tag{6.6}$$

The conformal factor φ^{-p} in Eq. (6.3) is eliminated by the corresponding factor coming from the volume factor of $\sqrt{\tilde{g}} = \sqrt{\exp(2f)g} = \varphi^p\sqrt{g}$.[12]

Corollary 6.1. *Thus, Eq.* (6.5) is the exact higher dimensional analog of the two-dimensional path integral, *Eq.* (5.5), *and, hence, problems related to higher dimensional CFT are those of Riemannian*[13] *quantum gravity* [49]. *In this*

[12] This result should be understood as follows. We have $\tilde{g}_{ij} = \varphi^{p-2}g_{ij}$. From here, we have for determinants $\sqrt{\tilde{g}} = [\varphi^{p-2}]^{\frac{d}{2}}\sqrt{g} = \varphi^p\sqrt{g}$ in accord with Eq. (3.5).

[13] Sometimes called Euclidean quantum gravity.

paper we are not considering the pseudo-Riemannian case relevant for true Einsteinian gravity. It will be discussed in a companion publication [32].

The question immediately arises: If the path integral, Eq. (6.5), is such an analog, is there a higher dimensional analog of Eq. (5.6)? The answer is "yes", if the dimension of space is even, and "no" if the dimension of space is odd [48].

6.2. Lack of conformal anomaly in odd dimensions

In view of its importance, we would like to provide a sketch of the arguments leading to the answer "no" in dimension 3. Clearly, the same kinds of arguments will apply in any other odd dimension. In doing this, although we follow the arguments of Refs. [48,50], some of our derivations are original. We begin by assuming that there is a one-parameter family of metrics: $\hat{g}(x) = \exp(2xf)g$. Next, we define the operator δ_f via

$$\delta_f \Box_g = \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0} \Box_{\exp(2xf)g}. \tag{6.7}$$

Taking into account Eq. (6.2) we obtain explicitly

$$\delta_f \Box_g = -2f\Box_g \tag{6.8}$$

and, accordingly,

$$\delta_f \mathrm{e}^{-t\Box_g} = -t(\delta_f \Box_g)\mathrm{e}^{-t\Box_g}. \tag{6.9}$$

These results allow us to write for the zeta function[14]

$$\begin{aligned}\delta_f \zeta_{\Box_g}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t t^{s-1} \delta_f \mathrm{Tr}(\mathrm{e}^{-t\Box_g}) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t t^{s} \mathrm{Tr}(-2f\Box_g \mathrm{e}^{-t\Box_g}) \\ &= \frac{-2s}{\Gamma(s)} \int_0^\infty \mathrm{d}t t^{s-1} \mathrm{Tr}(f\mathrm{e}^{-t\Box_g}).\end{aligned} \tag{6.10}$$

The last line was obtained by performing integration by parts. Since for small t's it is known that, provided that $\dim\ker\Box_g = 0$,[15]

$$\mathrm{Tr}(f\mathrm{e}^{-t\Box_g}) \simeq \sum_{k=0}^{\infty} \left(\int_M f(x)u_k(x)\mathrm{d}vol\right) t^{k-d/2}, \tag{6.11}$$

where $\mathrm{d}vol = \sqrt{g}\mathrm{d}^d x$, as usual. Using this result in Eq. (6.10) produces

$$\begin{aligned}\delta_f \zeta_{\Box_g}(0) &= -\frac{2s}{\Gamma(s)} \left(\int_0^1 \mathrm{d}t t^{s-1} \mathrm{Tr}(f\mathrm{e}^{-t\Box_g}) + \int_1^\infty \mathrm{d}t t^{s-1} \mathrm{Tr}(f\mathrm{e}^{-t\Box_g})\right) \\ &= -\frac{2s}{\Gamma(s)} \left.\left(\sum_{k=0}^{\infty} \frac{\int_M f u_k \mathrm{d}vol}{s+k-d/2} + \int_1^\infty \mathrm{d}t t^{s-1} \mathrm{Tr}(f\mathrm{e}^{-t\Box_g})\right)\right|_{s=0}.\end{aligned} \tag{6.12}$$

Since for $s \to 0^+$ we have $(1/\Gamma(s)) \sim s$, the second (regular) term in brackets will become zero when multiplied by the combination $\frac{2s}{\Gamma(s)}$, while the first term will become zero even if it might acquire a pole (in the case where $d = 2k$).

14 For example, see Eq. (5.8) and take into account the identity $x^{-s}\Gamma(s, x) = \int_0^\infty \mathrm{d}y y^{s-1} \exp(-xy)$.

15 In the case of path integrals this is always assumed since zero modes of the corresponding operators are associated with some kind of translational, rotational, etc. symmetry. To eliminate the undesirable dilatational symmetry, one actually should use the Yamabe functional, Eq. (3.6), as explained in Section 3. Although this is silently assumed thus far, arguments additional to those in Section 3 will be introduced further below.

Hence, for all dimensions $d \geq 3$ we obtain $\delta_f \zeta_{\Box_g}(0) = 0$ or

$$\zeta_{\Box_g}(0) = \zeta_{\Box_{\hat{g}}}(0). \tag{6.13}$$

This result can be used further. Indeed, if we write

$$\delta_f \left[\Gamma(s)\zeta_{\Box_g}(s)\right] = \Gamma(s)[\delta_f \zeta_{\Box_g}(0) + s\delta_f \zeta'_{\Box_g}(0) + O(s^2)] \tag{6.14}$$

and take into account Eq. (6.13) and the fact that $s\Gamma(s) = 1$ we obtain

$$\begin{aligned}\delta_f \zeta'_{\Box_g}(0) &= \delta_f \int_0^\infty \mathrm{d}t t^{s-1} \mathrm{Tr}(\mathrm{e}^{-t\Box_g}) \,|_{s=0} \\ &= -2s \left(\sum_{k=0}^{\infty} \frac{\int_M f u_k \mathrm{d}vol}{s+k-d/2} + \int_1^\infty \mathrm{d}t t^{s-1} \mathrm{Tr}(f \mathrm{e}^{-t\Box_g}) \right) \Bigg|_{s=0}.\end{aligned} \tag{6.15}$$

Applying the same arguments to Eq. (6.15) as were applied to Eq. (6.12) we conclude that, provided $\dim\ker \Box_g = 0$, in *odd* dimensions, $\delta_f \zeta'_{\Box_g}(0) = 0$, that is

$$\zeta'_{\Box_g}(0) = \zeta'_{\Box_{\hat{g}}}(0). \tag{6.16}$$

In view of Eq. (5.9) this leads also to

$$\det \Box_g = \det \Box_{\hat{g}}. \quad \Box \tag{6.17}$$

6.3. *3D CFT path integrals*

6.3.1. *General remarks*

Our previously obtained results can be further refined if we recall Eqs. (5.9)–(5.11). In particular, let $\mathrm{e}^{2\bar{\varphi}}$ in Eq. (5.11) be rewritten as some nonnegative constant l. Then we obtain

$$\zeta_{\hat{g}}(s) = l^s \zeta_g(s). \tag{6.18}$$

This result is consistent with Eq. (6.13) for $s = 0$. Differentiation with respect to s produces

$$\zeta'_{\hat{g}}(0) = \zeta_g(0) \ln l + \zeta'_g(0). \tag{6.19}$$

In view of Eq. (5.9), this result is equivalent to $\ln \det \Box_{\hat{g}} = \ln \det \Box_g - \zeta_g(0) \ln l$ and apparently contradicts Eq. (6.17). But the contradiction is only apparent in view of the earlier footnote. The situation is easily correctable if in the path integral, Eq. (6.5), we replace the action functional $S_\Box(\varphi)$ by that of Yamabe given by Eq. (3.2) (or (3.6)). This, by the way, allows us also to fix the value of $\zeta_g(0)$: provided that we identify the constant l with the volume V, the value of $\zeta_g(0) = \frac{2}{p}$. After this, the situation in the present case becomes very similar to that encountered in the previous section. That is, instead of the functional $\mathcal{F}(\varphi)$ given by Eq. (5.15), we can consider now the related functional given by

$$\begin{aligned}\mathcal{F}(\varphi) &= \ln \det \Box_g - \zeta_g(0) \ln V \\ &\equiv \ln \det \Box_g - \frac{2}{p} \ln V.\end{aligned} \tag{6.20}$$

For the path integral calculations, a functional defined in such a way is not yet sufficient. To repair this deficiency we have to impose a volume constraint. That is we need to consider the path integral of the type

$$Z_Y(V) = \int D[\varphi]\delta \left(\int_M \mathrm{d}^d x \sqrt{g} \varphi^p - V \right) \exp(-S[\varphi]) \tag{6.21}$$

with $S[\varphi]$ given by Eq. (3.2) (or (3.6)).

Remark 6.2. The path integral $Z_Y(V)$ (Y in honor of Yamabe) is the exact higher dimensional analog of the "stringy" path integral $S_L(A)$ given by Eq. (5.32). In view of Eq. (3.6), it also can be viewed as the path integral for pure (Euclidean) gravity in the presence of the cosmological constant.[16]

In complete analogy with Eq. (5.31), the standard path integral for the self-interacting scalar φ^4 (or LGW) field theory is obtainable now as follows:

$$\int D[\phi]\exp\{-S_{LGW}(\phi)\} = \int_0^\infty \mathrm{d}V \mathrm{e}^{-bV} Z_Y(V). \tag{6.22}$$

But, since the variation of the Yamabe functional produces the same Eq. (3.3) as can be obtained with help of the L–G functional, $S_{L\text{–}G}(\phi)$ (which, in view of the Remark 3.2, can be identified with $\tilde{S}[\varphi]$ defined by Eq. (3.7))[17], one can develop things differently but, surely, equivalently.

To this purpose, instead of the functional $S[\varphi]$ given in Eq. (3.2) we use

$$S_V[\varphi] = \frac{1}{V^{\frac{2}{p}}}\int_M \mathrm{d}^d x\sqrt{g}\{(\nabla_g \varphi)^2 + R(g)\varphi^2\} \tag{6.23}$$

and replace $S[\varphi]$ in the exponent of the path integral in Eq. (6.21) by $S_V[\varphi]$ from Eq. (6.23). Then, instead of Eq. (6.22), we obtain

$$\int D[\phi]\exp\{-S_{L\text{–}G}(\phi)\} \doteqdot \int_0^\infty \mathrm{d}V Z_Y(V), \tag{6.24}$$

where the sign $\doteqdot$ means "supposedly". This is so, because, in view of the Remark 3.3, at the level of saddle point calculations the left hand side and the right hand side produce the same G–L equation. Beyond the saddle point, calculations are not necessarily the same. Although we plan to discuss this issue in subsequent publications, some special cases are further discussed below in this section.

Remark 6.3. It should be clear that at the level of the saddle point calculation, replacement of the functional $S[\varphi]$ in Eq. (6.21) by $\mathcal{F}(\varphi)$ from Eq. (6.20) is completely adequate, so that the sequence of steps in the analysis performed for the two-dimensional case in Section 5 is transferable to higher dimensions without change.

Remark 6.4. Although in three dimensions we have the result given by Eq. (6.17), which forbids the use of identities like that in Eq. (5.6) still, on the basis of arguments just presented, the functional $\mathcal{F}(\varphi)$ defined by Eq. (6.20) should be used in the exponent of the corresponding path integral replacing that given in Eq. (5.15) in two dimensions. Since by doing so one will be confronted with the same kinds of minimization problems as were discussed earlier in Section 5,[18] the functional integral thus defined is an exact three-dimensional analog of the path integral, Eq. (5.29).

6.3.2. A sketch of 3D CFT calculations

Our results allow us to proceed in complete accord with Section 5. Thus, let $\hat{g}(u) = \mathrm{e}^{\phi(u)}g$ be a one-parameter family of metrics of fixed volume and such that $\hat{g}(0) = g$. This implies that $\phi(0) = 0$ and $\int_M \mathrm{e}^{\phi(u)}\mathrm{d}V_0 = V$. In two dimensions, using results of OPS [41], especially their Eq. (1.12), it is straightforward to obtain the following result:

$$\frac{\mathrm{d}}{\mathrm{d}u}(-\ln\det{}'\,\Delta_{g(u)})\,|_{u=0} = \dot{\zeta}'_g(0) = \frac{1}{12\pi}\int_M \dot{\phi}K(g)\mathrm{d}V_g, \tag{6.25}$$

[16] This fact can be used (with some caution in view of Remark 4.3) as an alternative formulation of the quantum gravity problem.

[17] To be in accord with standard texts on statistical mechanics, e.g. see [17], the constant scalar curvature $R(g)$ should be identified with the squared "mass", i.e. with $m^2 = a|T - T_c|$ where T and T_c are respectively the temperature and critical temperature. Also the sign "−" in front of the $\tilde{\lambda}$ term should be replaced by "+"

[18] This conclusion was reached without any reliance on path integrals and on physical applications in Ref. [51]. In this and related Ref. [20] the extremal properties of determinants of $\square_g$ and Δ_g with respect to variations of the background metric were studied.

where $K(g)$ is the Gaussian curvature for the metric g. $\dot{\zeta}_g(0)$ and $\dot{\phi}$ represent $\frac{\mathrm{d}}{\mathrm{d}u}\zeta_{g(u)}(0)\mid_{u=0}$ and $\frac{\mathrm{d}}{\mathrm{d}u}\phi(u)\mid_{u=0}$ respectively. Volume conservation implies

$$\frac{\mathrm{d}}{\mathrm{d}u}\int_M \mathrm{e}^{\phi(u)}\mathrm{d}V_g\mid_{u=0} = \int_M \dot{\phi}\mathrm{d}V_g = 0 \tag{6.26}$$

in accord with the earlier result, Eq. (5.17). If the Gaussian curvature $K(g)$ is constant, then Eqs. (6.25) and (6.26) produce the same result. This implies that $\frac{\mathrm{d}}{\mathrm{d}u}\zeta_{g(u)}(0)\mid_{u=0} = 0$, that is g is the "critical" (extremal) metric. In view of Eq. (6.25) this also means that, for such a metric, the free energy (e.g. see definition given by Eq. (5.5)) attains its extremum. By combining Eqs. (6.10), (6.14) and (6.15) we obtain at once

$$\begin{aligned}\delta_f\zeta'_{\Box_g}(0) &= \delta_f\int_0^\infty \mathrm{d}t t^{s-1}\mathrm{Tr}(\mathrm{e}^{-t\Box_g})\mid_{s=0}\\ &= \int_0^\infty \mathrm{d}t t^{s}\mathrm{Tr}(-2f\Box_g\mathrm{e}^{-t\Box_g})\mid_{s=0}.\end{aligned} \tag{6.27}$$

From here we obtain essentially the same result as the main theorem of Richardson [20], i.e.

$$\dot{\zeta}'_{\Box_{\hat{g}}}(0)\mid_{u=0} = 0 = \int_M \mathrm{d}V_g\dot{\phi}(x)\Box_g\zeta(1,x,x). \tag{6.28}$$

That is, provided that we replace $\Box_g$ by Δ_g and require that *locally* $\zeta(1,x,x) = \text{const}$, with the heat kernel $\zeta(s,x,x)$ given (as usual) by

$$\zeta(s,x,x) = \sum_{k=1}^{\infty}\frac{\psi_k^2(x)}{\lambda_k^s}, \tag{6.29}$$

we obtain the main result of Richardson, Ref. [20]; e.g. see his Theorem 1 and Corollary 1.1. Here $\psi_k(x)$ are eigenfunctions of the Laplacian (or Yamabe Laplacian) corresponding to eigenvalues λ_k with $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots$. In the two-dimensional case, the condition for criticality: $\dot{\zeta}'_g(0) = 0$ is *local*, meaning that, provided the volume is constrained, the constancy of the Gaussian curvature $K(g)$ at a given point of M is caused by the metric g for which $\dot{\zeta}'_g(0)$ is extremal. In three and higher dimensions, constancy of the curvature at the point of M is being replaced by constancy of $\zeta(1,x,x)$ under the same conditions of volume conservation. Such a condition is only necessary but is now not sufficient, since the analog of Moser–Trudinger inequality, Eq. (5.21), used to prove sufficiency in two dimensions does not exist. Instead, one should study locally the second variation of $\zeta'_{\hat{g}}(0)$ with respect to the underlying background metric in order to decide whether such a (local) extremum is a maximum or minimum. Fortunately, this task was accomplished in Refs. [20,51]. In particular, Richardson [20] obtained the following theorem.

Theorem 6.5 (*Richardson*). *The Euclidean metric on a cubic* 3*-torus is a local maximum of determinant of the Laplacian with respect to fixed volume conformal variations of the metric.*

Remark 6.6. This theorem is proven only for the cubic 3-torus. The word "local" means that there could be (or, there are, as we shall demonstrate) other 3-tori also providing local maxima for determinants. In fact, according to the result of Chiu [52], all determinants of flat 3-tori possess local maxima so that the determinant for the face centered cubic (fcc) lattice has the largest determinant. Physical implications of this fact can be found in our recent work, Ref. [47], and also will be briefly considered in Section 8.

Remark 6.7. By using the same arguments as in Hawking's paper, Ref. [53], the above theorem of Richardson can be given a physical interpretation. It is based on use of the saddle point methods applied to the Yamabe path integral, Eqs. (6.21) and (6.24).

The second variation of the Yamabe functional was calculated by Muto [54] (see also Ref. [49]) with the result

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{R}(g(t))\right)_{t=0} = \frac{d-2}{2}\left[\int_M \mathrm{d}V_g(\sigma(\nabla_g\varphi)^2 - R(g)\varphi^2)\right], \tag{6.30}$$

where the constant $\sigma = d - 1$. As in the case of quadratic actions in the flat space [17,36] the second variation (with volume constrained to be equal to 1) looks very much the same as the original quadratic Yamabe functional, except for the "wrong" sign in front of the scalar curvature. This "wrong" sign has important physical significance in both statistical mechanics and high energy physics. While the significance to high energy physics will be discussed in the companion publication, Ref. [32], here we would like to comment on the significance to statistical mechanics. For this we need to recall that for the G–L free energy functional $\mathcal{F}\{\varphi\}$ given by [17]

$$\mathcal{F}\{\varphi\} = \mathcal{F}_0 + \frac{1}{2}\int \mathrm{d}\mathbf{x}\left\{c\,(\nabla\varphi(\mathbf{x}))^2 + a\varphi^2(\mathbf{x}) + \frac{b}{2}\varphi^4(\mathbf{x})\right\} \tag{6.31}$$

the second variation is given by

$$\mathcal{F}\{\varphi\} = \mathcal{F}\{\varphi_0\} + \frac{1}{2}\int \mathrm{d}V \int \mathrm{d}V' \left[\frac{\delta^2}{\delta\varphi(\mathbf{x})\delta\varphi(\mathbf{x}')}\mathcal{F}\{\varphi\}\bigg|_{\substack{\varphi=\varphi_0\\ h=0}}\right]\eta(\mathbf{x})\eta(\mathbf{x}') + \cdots, \tag{6.32}$$

where

$$\frac{\delta^2}{\delta\varphi(\mathbf{x})\delta\varphi(\mathbf{x}')}\mathcal{F}\{\varphi\}\,|_{\varphi=\varphi_0} = \left(-c\,\nabla^2 + a + 3b\varphi_0^2\right)\delta(\mathbf{x}-\mathbf{x}') \tag{6.33}$$

with φ_0 being obtained from the extremum condition

$$\frac{\delta\mathcal{F}\{\varphi\}}{\delta\varphi(\mathbf{x})} = -c\,\nabla^2\varphi + a\varphi + b\varphi^3 - h = 0. \tag{6.34}$$

In a special case of a coordinate-independent field φ_0, Eq. (6.34) is reduced to

$$a\varphi_0 + b\varphi_0^3 = 0. \tag{6.35}$$

For $a < 0$ we obtain: $\varphi_0 = \pm(|a|/b)^{\frac{1}{2}}$, and $\varphi_0 = 0$. The solution $\varphi_0 = 0$ is not a minimum for the free energy and, hence, should be discarded. Of the two other solutions, the system chooses one of them (which is interpreted in the literature as *spontaneous symmetry breaking*). They both have the same free energy.[19] If we let now $\varphi(\mathbf{x}) = \varphi_0 + \delta\varphi \equiv \varphi_0 + \eta(\mathbf{x})$ and Taylor series expand $\mathcal{F}\{\varphi\}$ the result will coincide with Eqs. (6.32) and (6.33). Notice for $a < 0$ a combination $|a| + 3b\varphi_0^2 = 2\,|a| \equiv m^2$ (as compared to a in the high temperature phase). So, if in the high temperature phase $m^2 = R(g)$ with $R(g) > 0$, in the low temperature phase (i.e. below the criticality) $R(g) < 0$! Moreover, by choosing $d = 4$ in Eq. (6.30) we obtain a combination: $m^2 = \frac{4-2}{2}R(g) = 2R(g)$ in accord with the standard G–L results. The constant c in Eq. (6.33) is left unspecified in G–L theory while now it is equal to 6 (for $d = 4$). The spaces of constant negative curvature are hyperbolic.

Remark 6.8. In our earlier work, Ref. [55], we emphasized the major importance of hyperbolicity in statistical mechanical calculations related to the AdS/CFT correspondence. In this work we arrived at the same conclusions quite independently.

From the work by Muto [54] we know that: (a) if $R(g)$ is positive, the second variation can be made positive for appropriately chosen φ,[20] (b) if $R(g)$ is negative, the second variation is positive for the same reasons. The positivity of the second variation implies that the extremal *constant curvature* metric g provides a *locally stable minimum* for $\mathcal{R}(g(t))$ defined by Eq. (4.9). That is, using results of Section 4, the Einstein metric obtained as a solution to Eq. (4.2) is stable among nearby metrics. This conclusion will be reinforced in the next (dynamical) section.

Remark 6.9. It is interesting to notice that the calculation of higher order fluctuation corrections to the Yamabe path integral, Eq. (6.21), involves calculations on the moduli space of Einsteinian metrics, Ref. [56]. This observation

[19] Once the choice is made, the order parameter can be considered in all subsequent calculations as positive.

[20] This can be easily understood if we expand φ into Fourier series made of eigenfunctions of the Laplacian and take into account that for *any* closed Riemannian manifold the spectrum of the Laplacian is nonnegative and nondecreasing [29].

provides a strong link between higher dimensional LGW theory and two-dimensional string inspired CFT's discussed in the previous section. Naturally, Eq. (6.24) can be used to investigate to what extent the final results of conventional field-theoretic calculations, Refs. [17,36], might differ from more sophisticated string-theoretic calculations in the style of Refs. [14,15,46]. Additional details can also be found in our work, Ref. [47].

7. Critical dynamics and the Yamabe and Ricci flows

7.1. Physical motivation

The results obtained in previous sections are somewhat unrealistic from a purely practical (physical) point of view. The situation in our case is analogous to that known in thermodynamics. Recall that this discipline emerged from practical needs to improve the efficiency of heat engines. Clearly, rigorously speaking, it is not applicable to such devices since, by definition, it is valid only for time independent phenomena. Hence, it cannot be used at all because it usually takes a very long (if not infinite) time for the system to equilibrate, especially near the critical temperature T_c. Thus, the description of phase transitions using the G–L theory is valid only in a rather narrow range of temperatures around T_c. To bring a physical system into this range of temperatures (under constant pressure) requires varying the temperature of the surrounding environment in time. In addition, by definition, the "true"phase transition should take place only in the thermodynamic limit (of infinite volume with particle density kept constant). Since in the real world the systems under study are always of finite size, this requirement is implemented by imposing physically appropriate boundary conditions, e.g. periodic. Use of some appropriate boundary conditions causes systems undergoing phase transitions to actually "live" on some manifolds/orbifolds. Under such circumstances the topology and physics become intertwined. The signature of such boundary effects can be seen already in the calculation of determinants discussed in Sections 5 and 6. More on this subject is discussed in our earlier work, Ref. [55], where, in accord with Remark 6.8, we emphasized the role of hyperbolic spaces in the theory of phase transitions.[21]

The equilibration process known in the physics literature as *critical dynamics* can take a very long time (infinite in the thermodynamic limit for $T \rightarrow T_c$). Since idealization of reality is typical in physics, in this work we adopt the pragmatic (physical) point of view by considering systems of finite size with appropriately chosen boundary conditions. Then, the results of the previous sections are limiting cases of more general time dependent G–L theory considered for such systems.

Development of time dependent G–L theory was initiated by Landau and Khalatnikov in 1954 [57] and is also phenomenological. As such, it is based on the assumption that an order parameter φ satisfies the relaxation equation of the type given by

$$\frac{\partial \varphi}{\partial t} = -\gamma \frac{\delta \mathcal{F}\{\varphi\}}{\delta \varphi} \equiv -\gamma \,\mathrm{grad}\, \mathcal{F}\{\varphi\} \tag{7.1}$$

with functional $\mathcal{F}\{\varphi\}$ defined by Eq. (6.31) and the "friction" coefficient γ is some (assumed to be) known nonnegative constant. By rescaling time it can be eliminated. Such rescaling is assumed in all calculations below. Since such an equation was postulated, its validity was checked by real experiments with an excellent outcome [58]. Being armed with such results, we would like to develop the mathematical formalism of the previous sections to account for the effects of critical dynamics.

7.2. Mathematical motivation

Eq. (7.1) is an example of the gradient flow. In this subsection we would like to place the Landau–Khalatnikov theory of critical dynamics on a more rigorous mathematical basis involving the notion of the gradient flow.

We begin with reviewing the results of OPS, Ref. [41] (obtained for two-dimensional manifolds without boundaries), in the style of Hamilton's work, Ref. [16]. To this purpose we notice that Eq. (5.19) is an extremum of the functional $\mathcal{F}(\psi)$, Eq. (5.14). Clearly, if we write $\varphi = \psi + \varepsilon h$ in this functional so that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}(\psi+\varepsilon h)\,|_{\varepsilon=0} = -\int_M \mathrm{d}A_g \left(\Delta_g \psi\right) h + \frac{1}{2}\int_M \mathrm{d}A_g R(g) h - 2\pi\chi(M)\frac{\int_M \mathrm{d}A_g h \exp(2\psi)}{\int_M \mathrm{d}A_g \exp(2\psi)} \tag{7.2}$$

[21] Remark 6.8 is not in contradiction with the Remark 6.6. This is explained in great detail in our recent work, Ref. [47], and is also discussed in Section 8.

then, following OSP, the scalar product $\langle , \rangle$ in the tangent space to each point at which ψ is defined can be introduced according to convention:

$$\langle \operatorname{grad} \mathcal{F}, h\rangle \equiv (\delta\mathcal{F})\, h = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}(\psi + \varepsilon h)\,|_{\varepsilon=0}\,. \tag{7.3}$$

In accord with Eq. (3.6) of OPS, Ref. [41], this convention implies that

$$\operatorname{grad} \mathcal{F}\{\psi\} = -\Delta_g \psi + \frac{1}{2}R(g) - \frac{2\pi\,\chi(M)\exp(2\psi)}{\int_M \exp(2\psi)\mathrm{d}A_g}. \tag{7.4}$$

This result is to be compared with Eq. (5.19). Naturally, Eq. (5.19) corresponds to the condition of equilibrium, i.e. $\operatorname{grad} \mathcal{F}\{\psi\} = 0$. Away from equilibrium the dynamical equation *postulated* by OPS (see below) *coincides* with Eq. (7.1) postulated much earlier by Landau and Khalatnikov. To see this, we need to take into account Eqs. (5.2), (5.19) and (5.20). With their help we would like to rewrite Eq. (7.4) in the following *equivalent* form:

$$-\operatorname{grad} \mathcal{F}\{\psi\} = \frac{1}{2}(\langle \hat{R}(\hat{g})\rangle - \hat{R}(\hat{g}))\mathrm{e}^{2\psi}, \tag{7.5}$$

where $\langle \hat{R}(\hat{g})\rangle = (\int_M \mathrm{d}\hat{A}\hat{R}(\hat{g}))/\int_M \mathrm{d}\hat{A}$ with $\mathrm{d}\hat{A}$ given by Eq. (5.3). Accordingly, in view of Eq. (7.1), *up to time rescaling*, the equation describing critical dynamics can be written as follows:

$$\frac{\partial\psi}{\partial t} = \langle R(g)\rangle - R(g) \tag{7.6}$$

coinciding with Eq. (3.9) of OPS.[22]

Suppose that the equilibrium (the fixed point) condition,

$$\langle R(g)\rangle - R(g) = 0, \tag{7.7}$$

is valid in dimensions higher than 2 as well. Then it coincides formally with Eq. (2.12) defining scalar curvature or with Eq. (4.12) where it was obtained from the Hilbert–Einstein functional for pure gravity in the presence of the cosmological term. In view of this observation, we would like to demonstrate that equations like Eq. (7.6) whose fixed points are determined by the equations of the type given by Eq. (7.7) can indeed be obtained in higher dimensions. In this work only three-dimensional results will be discussed in some detail.

7.3. Critical dynamics in dimensions higher than 2: Yamabe versus Ricci flows

Following Hamilton [16], we begin with

Definition 7.1. The normalized *Ricci flow* is described by the dynamical equation given by

$$\frac{\partial}{\partial t}g_{ij} = \frac{2}{d}g_{ij}\langle R(g)\rangle - 2R_{ij}(g). \tag{7.8}$$

In writing this equation we use the notation of Section 4 where we defined $\langle R(g)\rangle = \left(\int_M R\sqrt{g}\mathrm{d}^d x\right)/\int_M \sqrt{g}\mathrm{d}^d x \equiv \left(\int_M R\mathrm{d}\mu\right)/\int_M \mathrm{d}\mu$. The above flow equation formally exists for $d \geq 2$ and should be considered along with some prescribed initial condition: $g_{ij}(t=0) = \hat{g}_{ij}$. In two dimensions it is always possible to write $R_{ij}(g) = \frac{1}{2}Rg_{ij}$ so that all two-dimensional spaces are Einsteinian [16]. Because of this fact, Eq. (7.8) is converted into

$$\frac{\partial}{\partial t}g_{ij} = (\langle R(g)\rangle - R(g))g_{ij}, \quad d = 2. \tag{7.9}$$

Definition 7.2. Eq. (7.9) defines the *Yamabe flow*. As such, it can be considered for any $d \geq 2$.

[22] In making such a comparison (as in Section 5) we took into account that $R = 2K$.

Remark 7.3. If, like in Section 2, we choose $\hat{g}_{ij} = e^{\psi} g_{ij}(0)$ and substitute this result into Eq. (7.9), we reobtain Eq. (7.6). Hence, Eq. (7.6) does describe the Yamabe flow.

Corollary 7.4. *Using results of the previous subsection (at least in two dimensions) the Yamabe flow is the* gradient *flow.*

Corollary 7.5. *Since the Yamabe flow in Eq. (7.9) is defined for any $d \geq 2$, the fixed points for such a flow should coincide with those given by Eq. (7.7).*

Remark 7.6. Only in two dimensions are the Yamabe and the normalized Ricci flows (essentially) equivalent. This is explained in great detail by Hamilton [16]. Since results of this paper crucially depend on the observation that *only* in two dimensions it is *always* possible to write $R_{ij}(g) = \frac{1}{2} R g_{ij}$, it should be clear that attempts to extend this equivalence to higher dimensions are destined for failure in those cases where the flow takes place on spaces which are not of Einstein type. Hence, in this case the task lies in finding conditions under which the Ricci flow, for which the initial metric is chosen to be not necessarily of Einstein type, leads to an Einstein-type metric(s) as the fixed point solution(s) for such a flow. This is discussed in the next subsection.

Even though Yamabe and Ricci flows are different in higher dimensions, one may still ask a related question: Can one use some Ricci flows in order to obtain results for Yamabe-type flows? We would like to demonstrate that this is indeed possible. To this purpose, using Eq. (7.9) we obtain

$$\frac{1}{2} g^{ij} \frac{\partial}{\partial t} g_{ij} = \langle R(g) \rangle - R(g). \tag{7.10}$$

In addition, since in any dimension $\frac{\partial}{\partial t} \mathrm{d}\mu \equiv \frac{\partial}{\partial t} \sqrt{\det g_{ij}}\, \mathrm{d}^d x = \frac{1}{2} g^{ij} \frac{\partial}{\partial t} g_{ij} \mathrm{d}\mu$, we obtain as well

$$\frac{\partial}{\partial t} \int_M \mathrm{d}\mu = \int_M \frac{\partial}{\partial t} \mathrm{d}\mu = \int_M (\langle R \rangle - R(g)) \mathrm{d}\mu = 0. \tag{7.11}$$

This equation is compatible with Eq. (7.8) and valid for $d \geq 2$. Indeed, if we take into account that $\frac{\partial}{\partial t} \mathrm{d}\mu = \frac{1}{2} g^{ij} \frac{\partial}{\partial t} g_{ij} \mathrm{d}\mu$, use Eq. (7.8) for $\frac{\partial}{\partial t} g_{ij}$, multiply both sides of this equation by $g^{ij} \mathrm{d}\mu$ and integrate the obtained result over M, we reobtain Eq. (7.11). Hence, after some manipulations, Eq. (7.8) describing Ricci flow (and valid for $d \geq 2$) can be brought into a form identical with Eq. (7.10) describing Yamabe flow (obtained originally for $d = 2$). This observation does not imply that the two flows are equivalent as Remark 7.6 indicates. Nevertheless, the Yamabe flow can be looked upon as some special case of the Ricci flow.

Corollary 7.7. *The arguments just presented demonstrate that solutions for the Yamabe flow can be obtained from solutions for the normalized Ricci flow. Moreover, in both cases the fixed points for such flows are given by Eq. (7.7) thus providing justification for the compatibility of two- and three-dimensional CFT discussed in previous sections and, hence, for the unique pathway for extending the results of 2 D CFT to three and higher dimensions.*

Corollary 7.8. *Since the fixed point, Eq. (7.7), is the traced form of the fixed point equation for the Ricci flow, Eq. (7.8), that is of the equation*

$$\frac{1}{d} g_{ij} \langle R(g) \rangle - R_{ij}(g) = 0, \tag{7.12}$$

which is equivalent to Eq. (4.2), this means: (a) *that the fixed point solutions of the normalized Ricci flow always produce the Einstein spaces, provided that these fixed points are stable and* (b) *that the arguments presented in Remark 4.3 are indeed valid.*

7.4. Stability of the fixed points solutions

The results obtained previously in this section cannot be used until the stability analysis of these solutions is performed. Since according to the Corollary 7.5 solutions for the Yamabe flow can be obtained from those for the Ricci flow, it is sufficient, in principle, to consider only the stability of the fixed point solutions for the Ricci flows. This path is not the most physically illuminating however as we would like to explain now.

7.4.1. Dynamics of the Yamabe flows

The Yamabe flow described by Eq. (7.9) should be supplemented with the initial condition, e.g. g_{ij} $(t = 0) = g_{ij}(0) \equiv \hat{g}$. In order to study the evolution of this metric it is convenient to write it in the following form [59]: $g_{ij}(t) = [\varphi(t)]^{\frac{4}{d-2}} g_{ij}(0)$. Using such a substitution in Eq. (7.9) (along with Eq. (2.12)) allows us to rewrite it in the following equivalent form:

$$\frac{\partial\varphi}{\partial t} = -\varphi^{2-p}(\alpha\Delta_{\hat{g}}\varphi + R(\hat{g})\varphi) + \langle R(g)\rangle\varphi \tag{7.13}$$

to be compared with the phenomenological result, Eq. (7.1), of Landau and Khalatnikov. Although these equations look somewhat different, their fixed points (if these are stable) are the same.[23] In view of the results of previous sections, we believe that the equations of critical dynamics known in the physical literature, e.g. see Ref. [58], all having their origin in the work by Landau and Khalatnikov [57], should be replaced by Eq. (7.13) also describing critical dynamics of physical systems of the Ginzburg–Landau type. Such a replacement is in accord with the requirement of conformal invariance at criticality as described in detail in our earlier work on AdS/CFT correspondence [55].

Remark 7.9. Evidently, physical realization of the Yamabe and Ricci flows[24] depends crucially on the possibility of such a replacement and, in view of results of this work, it can be considered as proven.

Using Eq. (7.10) we would like to introduce the following quantity:

$$\eta = \frac{4}{d-2}(\langle R(g)\rangle - R(g)). \tag{7.14}$$

In view of Eq. (7.11), integrating Eq. (7.14) we obtain

$$\int_M \eta \mathrm{d}\mu = 0. \tag{7.15}$$

By combining Eqs. (7.11) and (7.13), after some calculations which can be found in Ref. [59], we obtain

$$\frac{\partial}{\partial t}\langle R(g(t))\rangle = \frac{d-2}{2}\int_M R(g)\eta \mathrm{d}\mu. \tag{7.16}$$

Next, by combining Eqs. (7.14) and (7.16) we further obtain

$$\frac{\partial}{\partial t}\langle R(g(t))\rangle = 2\int_M R(g)(\langle R(g)\rangle - R(g)) = -2\int_M (\langle R(g)\rangle - R(g))^2, \tag{7.17}$$

where the second equality comes from the constraint, Eq. (7.15). The last expression possesses an entropic meaning. Indeed, following Boltzmann [60], we introduce the H-function (the entropy) for some gas made of hard spheres elastically scattering from each other. Explicitly, it is given by

$$H = -\int f \ln f \mathrm{d}\mathbf{v}, \tag{7.18}$$

where the distribution function $f(\mathbf{v}, t)$ obeys Boltzmann's equation

$$\frac{\partial}{\partial t} f = \int (f(\mathbf{v}', t)f(\mathbf{v}_1', t) - f(\mathbf{v}, t)f(\mathbf{v}_1, t))\,|\mathbf{v}_1 - \mathbf{v}|\, I(\theta, |\mathbf{v}_1 - \mathbf{v}|)\mathrm{d}\mathbf{v}_1\mathrm{d}\Omega \tag{7.19}$$

with $I(\theta, |\mathbf{v}_1 - \mathbf{v}|)$ being some known function, $\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$, and $\mathbf{v}' = \mathbf{v} + \mathbf{n}(\mathbf{n}\cdot\mathbf{g})$, $\mathbf{v}_1' = \mathbf{v}_1 - \mathbf{n}(\mathbf{n}\cdot\mathbf{g})$ being respective velocities of the colliding particles after (primed) and before (nonprimed) scattering on each other, where

[23] For two and three dimensions these fixed points were discussed in previous sections.

[24] And, ultimately, the physically assisted proof of the Poincaré and geometrization conjectures depend crucially on such an assumption.

$\mathbf{g} = \mathbf{v}_1 - \mathbf{v}$, and the vector $\mathbf{n}$ is a unit normal.[25] On the basis of these results, it can be shown that $\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{v}_1 = \mathrm{d}\mathbf{v}'\mathrm{d}\mathbf{v}'_1$. Using these results, let us consider calculation of $\frac{\partial}{\partial t}H$. By combining Eqs. (7.18) and (7.19) one obtains after some calculation $\frac{\partial}{\partial t}H \geq 0$ with equality only for the equilibrium case for which $f^*(\mathbf{v}', t)f^*(\mathbf{v}'_1, t) - f^*(\mathbf{v}, t)f^*(\mathbf{v}_1, t) = 0$. This condition is leading to the Maxwell's distribution of particle velocities, which produces a well tested equation of state for the ideal gas. In accord with thermodynamics, such defined entropy reaches its maximum at equilibrium.

Remark 7.10. The equilibrium result obtained for Maxwell's distribution depends crucially on the assumption of the occurrence of only binary collisions in the gas. Boltzmann's equation, Eq. (7.19), reflects just this fact. The results obtained become much less tractable if one wishes to account for ternary and higher order collisions. The same is true if one uses the full renormalization group flow equations [4] instead of just the leading order terms in these equations. The proof of Boltzmann's H-theorem and the proof of the Poincaré and geometrization conjectures are based on the validity of such approximations. The validity is assured by excellent agreement with experiment in the Boltzmann case and with equally good agreement with experiment in the case of Landau–Ginzburg critical dynamics described by the Yamabe-type flow.

On the basis of these physical arguments, we may choose $-\langle R(g(t))\rangle$ as our entropy (provided that the volume of the system is normalized, say, to unity). Then, we can identify Eq. (7.9) (or (7.13)) with a Boltzmann-type equation. Our previously obtained result, Eq. (4.12), then provides the physically meaningful equilibrium solution. Moreover, using these results we arrive at the conclusion that the fixed point solutions for the Yamabe flow are described by the Einstein equation, Eq. (4.2), leading to the Einstein manifolds of constant scalar curvature in accord with the Remark 4.3.

7.4.2. Dynamics of the Ricci flows

In this subsection we would like to demonstrate in some detail that some Ricci flows produce results that are indeed in agreement with those obtained for the Yamabe flows. For this purpose, following the logic of the previous subsection we have to find an analog of the entropy for the Ricci flows. This task was accomplished by Perelman in the first of his three papers, Ref. [1]. He noticed, albeit indirectly, that, unlike the Yamabe flow (which is the gradient-type flow), the Ricci flow is not of gradient type. The proof can be found in Ref. [61]. Accordingly, one cannot find an entropy for such a flow and, hence, study its stability. Perelman found an ingenious way around these difficulties. In view of the existing literature [2,3,10,61–63], our exposition will be brief, emphasizing the physical aspects of the results he obtained.[26]

7.4.2.1. The Boltzmann–Nash entropy and the dilaton gravity. In the case of the Yamabe flow our choice of $-\langle R(g(t))\rangle$ as an entropy seemed somewhat artificial in the sense that it was not motivated by any systematic procedure involving entropy as such. We would like to correct this deficiency in this subsection. We begin with the following

Definition 7.11. The heat operator $\square := \partial_t - \Delta_g$ has an operator $\square^* = -\partial_t - \Delta_g + R(g)$ as its adjoint with respect to the Ricci flow. For example, see Ref. [61], page 22.

Remark 7.12. The proof is based on the observation that upon some rescaling [5] it is possible to suppress the normalization in the Ricci flow, Eq. (7.8), that is to eliminate the factor $\frac{2}{d}g_{ij}\langle R(g)\rangle$. As result of this elimination, Eq. (7.10) acquires the form

$$\frac{1}{2}g^{ij}\frac{\partial}{\partial t}g_{ij} = -R(g) \tag{7.20}$$

used in the proof of the conjugacy of operators $\square$ and $\square^*$.

[25] See [60] for details.

[26] Naturally, only small parts of Perelman's results are discussed in this paper.

Let now g_{ij} be a solution of the nonnormalized Ricci flow equation and $u = \exp(-f)$ be the solution of the equation $\Box^* u = 0$ for some function f such that $\int_M \mathrm{e}^{-f} \mathrm{d}V = 1$. By analogy with Boltzmann's entropy, Eq. (7.18), following Ref. [61] we define the Nash entropy $N(u)$ as follows:

$$N(u) = \int_M u \ln u \mathrm{d}V. \tag{7.21}$$

The time derivative $\partial_t N(u)$ can be calculated now with the help of the adjoint equation $\Box^* u = 0$. Indeed, we obtain

$$\begin{aligned} \partial_t N(u) &= \int_M (\partial_t u \ln u \mathrm{d}V + \partial_t u \mathrm{d}V + u \ln u \partial_t \mathrm{d}V) \\ &= \int_M ((\partial_t - R(g)) u \ln u + \partial_t u) \mathrm{d}V \\ &= \int_M (-\Delta_g u \ln u + (R(g) - \Delta_g) u) \mathrm{d}V. \end{aligned} \tag{7.22a}$$

On a closed manifold, the integral of $\Delta_g u$ vanishes. Because of this, the last result acquires the form

$$\partial_t N(u) = \int_M \left(\frac{|\nabla_g u|^2}{u} + R(g) u \right) \mathrm{d}V = \int_M (|\nabla_g f| + R(g)) \mathrm{e}^{-f} \mathrm{d}V \equiv \mathcal{F}(g_{ij}, f). \tag{7.22b}$$

The functional $\mathcal{F}(g_{ij}, f)$ is Perelman's entropy [1] and is also an action for the dilaton gravity [8,64]. When f is a constant, taking into account the sign differences between the Boltzmann's entropy, Eq. (7.18), and the Nash entropy, Eq. (7.21), we obtain again our earlier result for the entropy $-\langle R(g(t)) \rangle$ as required.

Remark 7.13. The entropic nature of the Einstein–Hilbert and dilaton gravity actions just described might be especially useful for applications to black hole thermodynamics. Indeed, attempts to use the Ricci flow for the description of black hole dynamics were made recently in Ref. [65]. We plan to return to this issue in subsequent publications.

Remark 7.14. If one starts with the functional $\mathcal{F}(g_{ij}, f)$ without prior remarks on its entropic origin, one might employ the dilaton field f which is time independent. This leads to some technical simplifications to be discussed below.

Having defined the entropy for the Ricci flow, the task now lies in obtaining the analog of Eq. (7.17). We would like to do so in a way consistent with our results for the static case.

7.4.2.2. Steady solitons. In particular, following the logic of Section 7.2, we should begin with the replacement of the two-dimensional functional $\mathcal{F}(\psi)$ by its three-dimensional analog, e.g. by $\tilde{S}[\varphi]$ defined by Eq. (3.7), or, equivalently, by $S[\varphi]$ defined by Eq. (3.2). To make a connection with Perelman's work, we need temporarily to make a short but important detour. To this purpose we set $\varphi = \mathrm{e}^{-f/2}$ in Eq. (7.22). This causes the functional $\mathcal{F}(g_{ij}, f)$ to acquire the following new look:

$$\mathcal{F}(g_{ij}, \varphi) = \int_M (4 |\nabla \varphi|^2 + R(g) \varphi^2) \mathrm{d}V \tag{7.23}$$

to be compared with the functional $E[\varphi]$ defined in Eq. (3.2). By analogy with Eq. (3.4), now we can define the constant λ_g in terms of the Raleigh quotient, i.e.

$$\lambda_g = \inf_{\varphi} \frac{\mathcal{F}(g_{ij}, \varphi)}{\int_M \varphi^2 \mathrm{d}V}. \tag{7.24}$$

In accord with Eq. (3.3), the constant λ_g serves as an eigenvalue in the equation[27]

$$4 \Delta_g \bar{\varphi} + R(g) \bar{\varphi} - \lambda_g \bar{\varphi} = 0, \tag{7.25}$$

[27] See the second footnote for the sign convention for the Laplacian.

where $\bar{\varphi}$ is the minimizer for the Raleigh quotient, Eq. (7.24). Equivalently,

$$\lambda_g = \inf\left\{\int_M (4\,|\nabla\varphi|^2 + R(g)\varphi^2)\mathrm{d}V,\ \int_M \varphi^2 \mathrm{d}V = 1\right\}. \tag{7.26}$$

By analogy with Eqs. (4.7)–(4.10), we introduce the family of metrics $g_{ij}(s) = g_{ij} + sh_{ij}$ in order to consider the Raleigh quotient subject to such a variation. Then, instead of Eq. (4.10), we obtain[28]

$$\frac{\mathrm{d}}{\mathrm{d}s}\lambda(g_{ij}(s)) = \int_M (-h_{ij})(R_{ij} + \nabla_i\nabla_j f)\mathrm{e}^{-f}\mathrm{d}V. \tag{7.27}$$

Since this result is analogous to that given in Eq. (7.16), the course of action is going to be the same. In particular, following Perelman, and also [10], we notice that the variation of the functional $\mathcal{F}(g_{ij}, f)$ leads to the following coupled equations for the modified Ricci flow[29]:

$$\frac{\partial}{\partial t}g_{ij} = -2(R_{ij} + \nabla_i\nabla_j f), \tag{7.28a}$$

$$\frac{\partial}{\partial t}f = -R + \Delta f. \tag{7.28b}$$

Definition 7.15. The flow defined by these coupled equations is called the *generalized Ricci flow*.

Finally, after making the identification $\frac{\partial}{\partial t}g_{ij} = h_{ij}$ and using Eq. (7.28a) in Eq. (7.27) we obtain the monotonicity result of Perelman

$$\frac{\mathrm{d}}{\mathrm{d}s}\lambda(g_{ij}(s)) = 2\int_M \left|R_{ij} + \nabla_i\nabla_j f\right|^2 \mathrm{e}^{-f}\mathrm{d}V \tag{7.29}$$

to be compared with Eq. (7.17). In view of this comparison, we conclude that $\frac{\mathrm{d}}{\mathrm{d}s}\lambda(g_{ij}(s)) = 0$ only in the case when

$$R_{ij} + \nabla_i\nabla_j f = 0. \tag{7.30}$$

Definition 7.16. In the existing terminology, Eq. (7.30) is known as the equation for the *gradient steady soliton* [1–3, 66].

Following Perelman [part I] we can extend this result to the case of the so called breathers.

Definition 7.17. A metric $g_{ij}(t)$ evolving by the Ricci flow is called a *breather* if for some $t_1 < t_2$ and $\alpha > 0$ the metrics $\alpha g_{ij}(t_1)$ and $g_{ij}(t_2)$ differ only by a diffeomorphism. The cases when $\alpha = 1, \alpha < 1$ and $\alpha > 1$ correspond to *steady*, *shrinking* and *expanding* breathers respectively. *Solitons are trivial breathers* for which the above relationship between metrics holds for *any* pair of t_1 and t_2.

Corollary 7.18. *If one considers the Ricci flow as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms, then the breathers and the solitons are respectively the periodic orbits and fixed points for such a system.*

Corollary 7.19. *In view of the result, Eq.* (7.29)*, one can write* $\lambda(g_{ij}(s_1)) \leq \lambda(g_{ij}(s_2))$*. Because of Eq.* (7.30) *we obtain as well* $\lambda(g_{ij}(s_1)) = \lambda(g_{ij}(s_2))\ \forall s \in [s_1, s_2]$*. Hence, Eq.* (7.30) *providing a minimum for* $\lambda(g_{ij}(s))$ *is a steady breather and a steady soliton at the same time. That is, there are no breathers for the Ricci flows in accord with Perelman* [1].

28 For details of such calculations, please consult [10] and [66].

29 Please see again the sign convention for the Laplacian in the second footnote.

Going back to Eq. (7.30), multiplying it by g^{ij} and making a summation over repeated indices we obtain

$$R = \Delta_g f, \tag{7.31}$$

where, as before, we used the fact that $-\Delta_g f = \nabla^i \nabla_i f$. Taking into account arguments which lead from Eqs. (7.22a) to (7.22b) we conclude that for compact manifolds $f = \text{const}$ and, hence, $R = 0$.

7.4.2.3. Expanding and shrinking solitons. The obtained result is a special case of more general result of Ivey [67] who extended the work by Hamilton [16] for surface to Ricci flows on compact 3-manifolds.[30] It is instructive to conclude this subsection by discussing the physical significance of their results, especially in connection with results obtained in the previous section.

To this purpose, let us recall that the fundamental solution of the heat equation in d-dimensional Euclidean space can be written as [63]

$$u(x, y, \tau) = (4\pi\tau)^{-\frac{d}{2}} \exp(-|x-y|^2/4\tau), \tag{7.32}$$

where $\tau = t - T$ or $T - t = -\tau \equiv \sigma$ depending upon whether we are dealing with the forward or the backward heat equation.[31] If we let $y = 0$, then such a solution can be interpreted as a probability[32] since by design we have $\int_M u \mathrm{d}V = 1$. Using this fact, one can look for a solution of the heat equation on some Riemannian manifold M by employing the ansatz: $u = (4\pi\tau)^{-\frac{d}{2}} \mathrm{e}^{-f}$. This amounts of redefining the earlier introduced function $f : f = \tilde{f} - \frac{d}{2}\ln(4\pi\tau)$ so that $\mathrm{e}^{-\tilde{f}} = (4\pi\tau)^{-\frac{d}{2}} \mathrm{e}^{-f}$. Evidently, $\nabla f = \nabla \tilde{f}$ and, therefore, $\Delta_g f = \Delta_g \tilde{f}$ as well. To simplify matters, we would like to consider how things are done in the flat case. Using Eq. (7.32) we begin with the calculation of the Nash entropy. A simple calculation produces $N_{\text{flat}} = -\frac{d}{2} - \frac{d}{2}\ln(4\pi\tau)$. Using this result, we now define the properly normalized Nash entropy, i.e. $N(u) - N_{\text{flat}}$. Explicitly, we obtain

$$\tilde{N}(u) = N(u) - N_{\text{flat}} = \int_M \left(-f + \frac{d}{2}\right) u \mathrm{d}V. \tag{7.33}$$

Perelman does not explain how he obtained this result in his work. He calls such a normalized Nash entropy a "partition function"; e.g. see Section 5 of Part I of Ref. [1]. This is a bit misleading however because in the same section he earlier defines the partition function correctly, i.e. in accord with the accepted rules of statistical mechanics. According to these rules one defines the *free energy* $\mathcal{F}$ via $\ln \tilde{N}(u) \equiv -\beta\mathcal{F}$ with β being subsequently identified with the inverse temperature (provided that the system of units is used in which Boltzmann's constant $k_B = 1$). In the present case the role of temperature is played by τ. Using these definitions, one can define an "energy" $U = \langle E \rangle$ in a familiar way as $U = -\frac{\partial}{\partial\beta}\ln\tilde{N}(u) \equiv \tau\frac{\partial}{\partial\tau}\ln\tilde{N}(u)$. In our case, explicitly, we obtain

$$U = \tau^2 \frac{\partial}{\partial\tau}\left[\int_M u \ln u \mathrm{d}V - N_{\text{flat}}\right] = \int_M \tau^2 \frac{\partial}{\partial\tau}(u \ln u \mathrm{d}V) + \frac{d}{2}\tau \int_M u \mathrm{d}V. \tag{7.34a}$$

To proceed, we need to employ Eqs. (7.22a) and (7.22b).[33] This leads us to

$$U = \tau^2 \frac{\partial}{\partial\tau}\left[\int_M u \ln u \mathrm{d}V - N_{\text{flat}}\right] = \int_M \left[\tau^2(R + (\nabla f)^2) + \frac{d}{2}\tau\right] u \mathrm{d}V. \tag{7.34b}$$

This result differs slightly from that obtained by Perelman. If we are interested in calculation of the entropy, such a difference is useful. Indeed, since thermodynamically the relationship

$$\beta U - \beta\mathcal{F} = S \tag{7.35}$$

[30] Later, in Ref. [68], Hamilton came up with yet another proof of Ivey's results.

[31] Here T is some pre-assigned time.

[32] The fundamental solution, Eq. (7.32), also has the meaning of a probability in polymer physics [69]. This fact is briefly discussed in Section 8.

[33] Here we took into account that $\int_M u \mathrm{d}V = 1$ as required.

determines the entropy S, using results already obtained for U and $\mathcal{F}$ we obtain

$$S_+ = \int_M [\tau(R + (\nabla f)^2) - f + d]u\mathrm{d}V. \tag{7.36}$$

This result coincides with the result for the entropy of the Ricci expanders obtained in Ref. [61] where a considerably more cumbersome and lengthy pathway was chosen in order to obtain it. In order to obtain the entropy for the Ricci shrinkers, it is sufficient to change signs when taking time derivatives. Hence, we obtain at once

$$S_- = \int_M [\sigma(R + (\nabla f)^2) + f - d]u\mathrm{d}V, \tag{7.37}$$

again in accord with Ref. [61]. At this point, one can proceed either by computing the heat capacity $C_v = (\frac{\partial}{\partial T}U)_V$ under the constant volume or repeat the arguments for the steady solitons adapted to the present case. The last pathway is discussed in the paper by Cao, Ref. [66]. More physically attractive, however, is to follow the logic of Perelman and to take into account that[34]

$$C_v = \left(\frac{\partial}{\partial\beta}U\right)\frac{\partial\beta}{\partial\tau} = -\left(\frac{\partial^2}{\partial\beta^2}\ln\tilde{N}(u)\right)\frac{\partial\beta}{\partial\tau},$$

or

$$\tau^2 C_v = \frac{\partial^2}{\partial\beta^2}\ln\tilde{N}(u). \tag{7.38}$$

A straightforward but lengthy calculation analogous to that used in (7.34) finally leads to

$$\tau^2 C_v = \tau^4 \int_M \left| R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}\right|^2 \mathrm{d}V \tag{7.39}$$

in accord with earlier obtained result, Eq. (7.29), for the steady soliton (obtained from this expression in the limit $|\tau| \to \infty$). The gradient shrinking and expanding solitons are respectively solutions to

$$R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\sigma}g_{ij} = 0 \tag{7.40a}$$

and

$$R_{ij} + \nabla_i\nabla_j f + \frac{1}{2\tau}g_{ij} = 0. \tag{7.40b}$$

By analogy with Eq. (7.30) we multiply both of these equations by g^{ij} and sum over repeated indices in order to obtain

$$R - \Delta_g f - \frac{d}{2\sigma} = 0 \tag{7.41a}$$

and

$$R - \Delta_g f + \frac{d}{2\tau} = 0. \tag{7.41b}$$

For compact manifolds $\Delta_g f = 0$ as before. As a result, in both cases we obtain again Eq. (4.3). By combining these results with those which follow from Eq. (7.31) we thus have rederived the following result of Ivey.

[34] As is well known from statistical mechanics,

$$C_v = \beta^2(\langle E^2\rangle - \langle E\rangle^2),$$

where, as before, we put $k_B = 1$. Hence, what Perelman is simply calling a fluctuation in Section 5 of Part I of his work, Ref. [1], is actually a heat capacity.

Theorem 7.20. *There are no three-dimensional solitons or breathers on a compact connected* 3*-manifold other than those of constant curvature metrics.*

Remark 7.21. This theorem was originally proved by Ivey, Ref. [67], who was inspired by the earlier obtained result of Hamilton [16] for surfaces. Our derivation, however, is inspired by Perelman, Ref. [1], who uses physical arguments. Because these arguments are not presented in sufficient detail in his papers, they were left unappreciated in the works by other mathematicians [2,3,10,61–63].

Corollary 7.22. *In view of the results obtained it appears that, at least for compact connected* 3*-manifolds, one can extract the needed physical information for the Yamabe flow from that for the Ricci flow. We refer our readers to works by Hamilton* [5,16,70] *where many illuminating additional details can be found. The results obtained provide needed support to claims made in Remark* 7.6*.*

Corollary 7.23. *The generality of arguments used for re-proving the Theorem* 7.20 *are such that they can be applied, in principle, to manifolds of any dimension $d \geq 2$. Because of this, for compact manifolds, the Euclidean dilaton gravity described by the Eq.* (7.22b) *for any $d \geq 2$ is reduced to the more familiar Euclidean gravity.*

8. Discussion

8.1. Other physical processes whose dynamics is described by Ricci flows

The goal of this paper is to find some processes taking place in the real world providing a physical manifestation for Ricci flows. In the previous sections such processes were found, e.g. in the form of critical dynamics. The question arises of whether there are other physical processes which also can be used for justifying the existence of Ricci flows. Earlier, in the text, in Ref. [65], we mentioned the dynamics of black holes. In addition, we would like to note that the fundamental solution, Eq. (7.32), of the heat equation can be interpreted as an end-to-end distance distribution function used in polymer physics for the computation of various averages, e.g. the mean square end-to-end distance for the flexible polymer being modelled by a random walk [69]. Under such circumstances, the time τ is interpreted as a polymer's contour length. The curvature effects can be interpreted in terms of the polymer's backbone rigidity [71], etc. Upon Fourier–Laplace transform of the distribution function, Eq. (7.32), the propagator for the Klein–Gordon quantum field can be obtained [7,72]. Hence, it is possible to provide an interpretation of the Ricci flow processes in terms of some dynamics of quantum fields, etc. Whatever processes we may discuss, it should be clear that a specific physical process can be used only as some specific realization of the Ricci flow. Since such a situation is common for all real life processes described by the equations of mathematical physics, the task lies in finding different physical situations in which this flow can be realized. For instance, since conducting experiments on black holes [65] is unrealistic, one can think of analogous processes in liquid helium as described in the monograph by Volovik, Ref. [73]. A great number of additional condensed matter analogs of gravity-related phenomena can be found in Refs. [74] and [75] and references therein.

8.2. Interplay between topology, geometry and physics

To make our discussion complete, let us return to the processes involving critical dynamics studied in this paper. In view of Theorem 7.20, the scalar curvatures obtained either will go to zero (for expanding solitons) or will blow up (for shrinking solitons) when time $t \to \infty$ [2]. Physically, the case of curvatures blowing up is not acceptable, however; e.g. see Eq. (3.1) and comments related to this equation.[35] Hence, physically acceptable manifolds must either be flat or hyperbolic in accord with Eq. (6.30) and Remark 6.8. Although this fact was discussed in our earlier works, [47,55], we would like to provide a sketch of relevant arguments in this subsection.

We begin with two-dimensional CFT models at criticality following Ref. [21], pages 340–344, and Ref. [47], Section 5, where additional details can be found. Many of these models can be obtained by some straightforward

[35] It may be of some relevance to black holes, however. This topic requires further study within the context of liquid helium and other kinds of condensed matter experiments just mentioned.

modifications of the simplest Gaussian model defined on the torus T^2. If ω_1 and ω_2 are two toral periods, one can define their ratio τ as $\tau = \frac{\omega_2}{\omega_1}$. The number τ is necessarily complex with $\operatorname{Im}\tau = \left|\frac{1}{\omega_1}\right|^2 \equiv y$. In terms of this notation, the dimensionless free energy $\mathcal{F}$ for the Gaussian model is obtained as follows:

$$\mathcal{F} = \frac{1}{2} \ln y \, |\eta(\tau)|^2 , \tag{8.1}$$

where the function $\eta(\tau)$ is the Dedekind eta function. Such a function is known as the modular function for the once-punctured torus [76]. Thus, even though originally the torus topology for the Gaussian model was used, the actual computations for this model involving the use of the first Kronecker limit formula [47] lead to the final manifestly modular invariant result for $\mathcal{F}$. The price for modular invariance is the switch in topology: from that for a flat torus (before the Kronecker limit is taken) to that for a punctured torus (after the limit is taken). The punctured torus can be obtained from some parallelogram of periods in the complex plane $\mathbf{C}$ whose vertices are removed and whose sides are identified pairwise. Since Euler's characteristic χ for the torus is 0 while that for the punctured torus is -1, the punctured torus represents an example of a hyperbolic surface (actually an orbifold [77]) as discussed in great detail in our work Ref. [78]. Such a surface is not compact, however, while the results of the previous section were developed for compact surfaces.[36] In two dimensions the situation can be remedied by using the Schottky double construction widely used in designing of CFT models [79]. Use of this construction causes us to replace the punctured torus by the double torus, i.e. by the Riemann surface of genus 2 which is also hyperbolic.

As we demonstrated in [47] (see also Remark 6.6), even if one begins with the standard Euclidean cube (or any parallelepiped for that matter whose sides are pairwise identified thus making it a T^3), use of the analog of the first Kronecker limit formula in 3D leads to a result similar to Eq. (8.1) indicating that in complete accordance with the 2D case, the limiting manifold/orbifold is hyperbolic. This fact is consistent with results of our earlier work Ref. [55] on AdS/CFT correspondence where totally independent arguments were used to arrive at such a conclusion.

We conclude this work with a brief discussion of how one can actually design compact hyperbolic 3-manifolds. The symplest 3D hyperbolic manifold in which a 3D CFT lives and evolves can be built using a 2D punctured torus. To do so, we would like to use the results of our earlier work, Ref. [78], in which we noticed that such a punctured torus is the Seifert surface for the figure of eight (hyperbolic) knot. The evolution of surface automorphisms in fictitious time creates a specific hyperbolic 3-manifold known as a 3-manifold fibered over the circle (the puncture on the torus can be opened up so that it is homeomorphic to a circle S^1) is discussed in our work, Ref. [78]. The 3-manifold created in such a way will be cusped [55] and, hence, noncompact. To make such a manifold compact one can also use a hyperbolic double (analogous to a Schottky double as described in detail in Ref. [80]). Perelman's work is not limited to compact manifolds. The only trouble with noncompactness lies in the fact that the Yamabe flow considered in this work leads to Einstein-type spaces so that compactness is synonymous with being of Einstein type. Refs. [26,27] discuss situations in which Yamabe functionals for manifolds with boundary are being considered. More relevant to physical applications is the work by Mazzeo et al. [81] and also Ref. [82], in which the Yamabe problem was studied for noncompact manifolds. It would be very interesting and challenging to study the Yamabe flow for noncompact manifolds discussed in Refs. [81,82]. Many details of the construction of 3-manifolds and orbifolds are discussed in the exceptionally well written Ref. [77]. We conclude our paper by urging our readers to read this reference, which may be also helpful for developing a solid understanding of the Poincaré and geometrization conjectures.

Note added in proof

After this work was completed several important papers supporting and clarifying the results of our Section 8.2 came to our attention. In particular, in the paper by Long and Reid (Algebraic and Geometric Topology 2 (2002) 285–296) it is shown that all flat manifolds can be looked upon as cusps of hyperbolic orbifolds; e.g. see our works, Refs. [55,78] for a quick introduction to cusps and Ref. [77] for an introduction to hyperbolic orbifolds. Many additional details elaborating on the work by Long and Reid can be found in the Ph.D. thesis by David Ben McReinolds (arxiv:math.GT/0606571). Some related material can also be found in the paper by X. Dai (arxiv:math.DG/0106172).

[36] This restriction is not essential however. It is possible to extend the theory of Ricci and Yamabe flows to noncompact surfaces [1,2]. Such an extension requires the use of mathematical methods more sophisticated than those used in our paper.

References

[1] G. Perelman, The entropy formula for the Ricci flow and its geometric applications. arxiv:math.DG/0211159;
G. Perelman, Ricci flow with surgery on three-manifolds. arxiv:math.DG/0303109;
G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. arxiv:math.DG/0307245.
[2] H.-D. Cao, X.-P. Zhu, A complete proof of the Poincaré and geometrization conjectures—Application of the Hamilton–Perelman theory of the Ricci flow, Asian J. Math. 10 (2006) 165–492.
[3] J. Morgan, G. Tian, Ricci flow and the Poincaré conjecture. arxiv:math.DG/0607607.
[4] D. Friedan, Nonlinear models in $2 + \varepsilon$ dimensions, Ann. Phys. 163 (1985) 318–419.
[5] R. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geom. 17 (1982) 255–306.
[6] B. Chow, P. Lu, Lei Ni, Hamilton's Ricci Flow, AMS Publishers, Providence, RI, 2006.
[7] A. Polyakov, Gauge Fields and Strings, Harwood Academic Publ., New York, NY, 1987.
[8] M. Green, J. Schwarz, E. Witten, Superstring Theory, vol. 1, Cambridge U. Press, Cambridge, UK, 1987.
[9] D. Gross, J. Harvey, E. Martinec, R. Rohm, Heterotic string theory, Nuclear Phys. B 267 (1986) 75–124.
[10] B. Kleiner, J. Lott, Notes on Perelman's papers. arxiv:math.DG./06056667.
[11] L. Bessieres, Conjecture de Poincaré: la preuve de R. Hamilton et G. Perelman, SMF-Gazette 106 (2005) 7–35.
[12] M. Kapovich, Geometrization conjecture and the Ricci flow. www.math.ucdavis.edu/~kapovich/EPR/ricci.ps.
[13] B. Hatfield, Quantum Field Theory of Point Particles and Strings, Addison-Wesley Publ. Co., New York, NY, 1992.
[14] E. Abdalla, M. Abdalla, D. Dalmazi, A. Zadra, 2D-Gravity in Non-Critical Strings, Springer-Verlag, Berlin, 1994.
[15] Yu. Nakayama, Liouville field theory—A decade after revolution. arxiv:hep-th/0402009.
[16] R. Hamilton, The Ricci flow on surfaces, Contemp. Math. 71 (1988) 237–262.
[17] D. Amit, Field Theory, the Renormalization Group and Critical Phenomena, McGraw-Hill Inc., London, 1978.
[18] C. Itzykson, J.-B. Zuber, Quantum Field Theory, McGraw-Hill, London, 1980.
[19] J. Lee, T. Parker, The Yamabe problem, BAMS 17 (1987) 37–91.
[20] K. Richardson, Critical points of the determinant of the Laplace operator, J. Funct. Anal. 122 (1994) 52–83.
[21] P. Di Francesco, P. Mathieu, D. Senechal, Conformal Field Theory, Springer, Berlin, 1997.
[22] R. Wald, General Relativity, The University of Chicago Press, Chicago, 1984.
[23] A. Kholodenko, K. Freed, Theta point ("tricritical") region behavior for a polymer chain: Transition to collapse, J. Chem. Phys. 80 (1984) 900–924.
[24] L.D. Landau, E.M. Lifshitz, Statistical Physics. Part 1, in: Course in Theoretical Physics, vol. 5, Pergamon Press, Oxford, 1982.
[25] H. Bray, A. Neves, Classification of prime 3-manifolds with σ-invariant greater than RP^3, Ann. Math. 159 (2004) 407–424.
[26] J. Escobar, Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary, Ann. Math. 136 (1992) 1–50.
[27] K. Akutagawa, B. Botvinnik, Yamabe metrics on cylindrical manifolds, GAFA 13 (2003) 259–333.
[28] R.M. Schoen, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics, Lecture Notes Math. 1365 (1989) 120–184.
[29] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, Springer, Berlin, 1998.
[30] P. Dirac, General Theory of Relativity, Princeton U. Press, Princeton, 1996.
[31] H. Yamabe, On a deformation of a Riemannian structures on compact manifolds, Osaka Math. J. 12 (1960) 21–37.
[32] A. Kholodenko, Quantum gravity and QCD in the light of works by Hamilton and Perelman (in preparation).
[33] J. Schouten, Ricci-Calculus, Springer-Verlag, Berlin, 1954.
[34] K. Belov, About extrinsic geometry of hypersurfaces, Soviet. Math. Izvestiya 104 (1971) 1–8 (in Russian).
[35] C. Fedischenko, V. Chernyshenko, About some generalization of spaces of constant curvature, Tensor Anal. Appl. 11 (1961) 269–276 (in Russian).
[36] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Clarendon Press, Oxford, 1989.
[37] A. Polyakov, Quantum geometry of bosonic strings, Phys. Lett. B 103 (1981) 207–210.
[38] O. Alvarez, Theory of strings with boundary, Nucl. Phys. B 216 (1983) 125–184.
[39] W. Weisberger, Conformal invariants for determinants of Laplacians on Riemann surfaces, Comm. Math. Phys. 112 (1987) 633–638.
[40] W. Weisberger, Normalization of the path integral measure and coupling constants for bosonic strings, Nucl. Phys. B 284 (1987) 171–200.
[41] B. Osgood, R. Phillips, P. Sarnak, Extremals of determinants of Laplacians, J. Funct. Anal. 80 (1988) 148–211.
[42] B. Osgood, R. Phillips, P. Sarnak, Compact isospectral sets of surfaces, J. Funct. Anal. 80 (1988) 212–234.
[43] B. Osgood, R. Phillips, P. Sarnak, Moduli space, heights and isospectral sets of plane domains, Ann. Math. 129 (1989) 293–362.
[44] E. Onofri, On the positivity of the effective action in a theory of random surfaces, Comm. Math. Phys. 86 (1982) 321–326.
[45] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality, Ann. Math. 138 (1993) 213–242.
[46] M. Goulian, M. Li, Correlation functions in Liouville theory, Phys. Rev. Lett. 66 (1991) 2051–2055.
[47] A. Kholodenko, E. Ballard, From Ginzburg–Landau to Hilbert–Einstein via Yamabe, Physica A 380 (2007) 115–162.
[48] S. Rosenberg, The Laplacian on a Riemannian Manifold, Cambridge Univ. Press, Cambridge, UK, 1997.
[49] A. Besse, Einstein Manifolds, Springer-Verlag, Berlin, 1987.
[50] T. Parker, S. Rosenberg, Invariants of conformal Laplacians, J. Differential Geom. 25 (1987) 535–557.
[51] K. Okikiolu, Critical metrics for the determinant of the Laplacian in odd dimensions, Ann. Math. 153 (2001) 471–531.
[52] P. Chui, Height of flat tori, AMS Proc. 125 (1997) 723–730.
[53] S. Hawking, Zeta function regularization of path integrals in curved space, Comm. Math. Phys. 55 (1977) 133–148.

[54] Y. Muto, On Einstein's metrics, J. Differential Geom. 9 (1974) 521–530.
[55] A. Kholodenko, Boundary CFT, limit sets of Kleinian groups and holography, J. Geom. Phys. 35 (2000) 193–238.
[56] N. Koiso, Einstein metrics and complex structures, Invent Math. 73 (1983) 71–106.
[57] L. Landau, I. Khalatnikov, About anomalous sound attenuation near critical points of the phase transitions of second kind, Sov. Phys. Doklady 96 (1954) 469–472.
[58] P. Chaikin, T. Lubensky, Principles of Condensed Matter Physics, Cambridge U. Press, Cambridge, UK, 2000.
[59] H. Schwetlick, M. Struwe, Convergence of the Yamabe flow for "large" energies, J. Reine Angew. Math. 562 (2003) 59–100.
[60] A. Isihara, Statistical Physics, Academic Press, New York, NY, 1971.
[61] R. Müller, Differential Harnac Inequalities and the Ricci Flow, European Math. Soc. Publ. House, Zurich, SW, 2006.
[62] P. Topping, Lectures on the Ricci Flow, Cambridge U. Press, Cambridge, UK, 2006.
[63] A. Borisenko, An introduction to Hamilton's and Perelman's work on conjectures of Poincaré and Thurston. www-mechmath.univer.kharkov.ua/geometry/preprint/perelman7.pdf.
[64] Y. Fujii, K.-I. Maeda, The Scalar–Tensor Theory of Gravitation, Cambridge U. Press, Cambridge, UK, 2003.
[65] M. Headrick, T. Wiseman, Ricci flow and black holes. arxiv:hep-th/0606086.
[66] H.-D. Cao, Geometry of Ricci solitons, Chinese Ann. Math. 27B (2) (2006) 121–142.
[67] T. Ivey, Ricci solitons on compact three-manifolds, Differential Geom. Appl. 3 (1993) 301–307.
[68] R. Hamilton, Non-singular solutions of the Ricci flow on three-manifolds, Comm. Anal. Geom. 7 (1999) 695–729.
[69] P. De Gennes, Scaling Concepts in Polymer Physics, Cornell Univ. Press, Ithaca, NY, 1979.
[70] R. Hamilton, Formation of singularities in the Ricci flow, Surv. Differ. Geom. 2 (1995) 7–136.
[71] A. Kholodenko, Fermi–Bose transmutation: From semiflexible polymers to superstrings, Ann. Phys. 202 (1990) 186–225.
[72] A. Kholodenko, Statistical mechanics of deformable droplets on flat surfaces, J. Math. Phys. 37 (1996) 1287–1313.
[73] G. Volovik, The Universe in a Helium Droplet, Clarendon Press, Oxford, 2003.
[74] W. Zurek, Cosmological experiments in condensed matter systems, Phys. Rep. 276 (1996) 177–221.
[75] C. Barcelo, S. Liberati, M. Visser, Analogue gravity from field theory normal modes, Class. Quantum Grav. 18 (2001) 3595–3610.
[76] H. Iwaniec, Topics in Classical Automorphic Forms, AMS Publishers, Providence, RI, 1997.
[77] D. Cooper, C. Hodgson, S. Kerhoff, Three-Dimensional Orbifolds and Cone Manifolds, Japan Publ. Trading Co., Tokyo, 2000.
[78] A. Kholodenko, Statistical mechanics of $2+1$ gravity from Riemann Zeta function and Alexander polynomial: Exact results, J. Geom. Phys. 38 (2001) 81–139.
[79] C. Schweigerth, J. Fuchs, J. Walcher, Conformal field theory, boundary conditions and applications to string theory. arxiv:hep-th/0011109.
[80] K. Matsuzaki, M. Taniguchi, Hyperbolic Manifolds and Kleinian Groups, Clarendon Press, Oxford, 1998.
[81] R. Mazzeo, D. Pollack, K. Uhlenbeck, Moduli spaces of singular Yamabe metrics, AMS J. 9 (1996) 303–344.
[82] A. Chouikha, Ricci curvature and singularities of constant scalar curvature metrics. arxiv:math.DG/0501505.